ELEC system identification workshop

Optimization methods

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Plan

1. Behavioral approach
2. Subspace methods
3. Optimization methods

Outline

Approximation error–model complexity trade-off

System identification $\leftrightarrow$ low-rank approximation

Solution methods: variable projection

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Approximation error–model complexity trade-off

System identification $\leftrightarrow$ low-rank approximation

Solution methods: variable projection

An exact model contains the data
Any model that is not exact is approximate

$$w \subset \mathscr{B} \quad :\Longleftrightarrow \quad \text{"}\mathscr{B} \text{ is exact model for } w\text{"}$$

$$w \not\subset \mathscr{B} \quad :\Longleftrightarrow \quad \text{"}\mathscr{B} \text{ is approximate model for } w\text{"}$$

To compare approximations, we use criteria

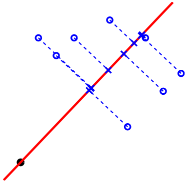
misfit criterion $\|w - \widehat{w}\|$

modify w as little as possible,
so that $\widehat{w}$ is exact

latency criterion $\|e\|$

augment $\mathscr{B}$ by as small as possible e,
so that (e, w) is exact

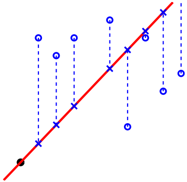
In the linear static case, misfit and latency lead to the TLS and OLS problems



misfit (total least squares)

$$\min_{\widehat{u},\widehat{y},\theta} \left\| \begin{bmatrix} u-\widehat{u} & y-\widehat{y} \end{bmatrix} \right\|_{\mathrm{F}} \quad \text{s.t.} \quad \underbrace{\widehat{u}\theta = \widehat{y}}_{(\widehat{u},\widehat{y})\subset\mathscr{B}(\theta)}$$

$\widehat{w} = (\widehat{u},\widehat{y})$ approximates $w = (u,y)$



latency (ordinary least squares)

$$\min_{\widehat{e},\theta} \|\widehat{e}\|_2 \quad \text{s.t.} \quad \underbrace{u\theta = y+\widehat{e}}_{(\widehat{e},u,y)\subset\mathscr{B}_{\mathrm{ext}}(\theta)}$$

$\widehat{e}$ is unobserved (latent) input

There is a one-to-one relation between noise model and approximation criterion

stochastic estimation $\leftrightarrow$ deterministic approximation

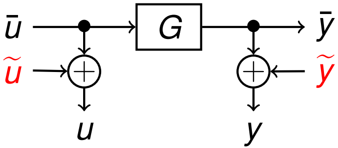
$$\text{noise model} \xleftrightarrow{\text{maximum likelihood}} \text{cost function}$$

also in control

LQG control $\leftrightarrow$ H_2 optimal control

In a stochastic setting, misfit and latency correspond to EIV and ARMAX problems

EIV $\leftrightarrow$ misfit

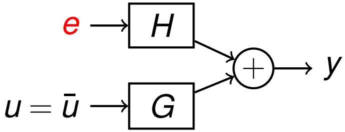


$\widetilde{u}, \widetilde{y}$ — measurement errors

$$\min_{\widehat{w} \subset \mathscr{B}} \; \| w - \widehat{w} \|$$

$$\mathscr{B} := \left\{ \begin{bmatrix} \widehat{u} \\ \widehat{y} \end{bmatrix} \mid \widehat{y} = \widehat{G}\widehat{u} \right\}$$

ARMAX $\leftrightarrow$ latency

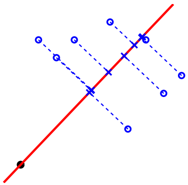


e — disturbance

$$\min_{(\widehat{e}, w) \subset \mathscr{B}_{\text{ext}}} \; \| \widehat{e} \|$$

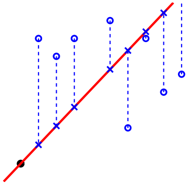
$$\mathscr{B}_{\text{ext}} := \left\{ \begin{bmatrix} \widehat{e} \\ u \\ y \end{bmatrix} \mid y = \begin{bmatrix} \widehat{H} & \widehat{G} \end{bmatrix} \begin{bmatrix} \widehat{e} \\ u \end{bmatrix} \right\}$$

Summary: approximation criterion



TLS $\leftrightarrow$ misfit $\leftrightarrow$ errors-in-variables

$$\min_{\widehat{w} \subset \mathscr{B}} \|w - \widehat{w}\| \quad \left(\begin{array}{c} \text{projection} \\ \text{of } w \text{ on } \mathscr{B} \end{array} \right)$$



OLS $\leftrightarrow$ latency $\leftrightarrow$ ARMAX

$$\min_{(\widehat{e}, w) \in \mathscr{B}_{\text{ext}}} \|\widehat{e}\|$$

A general problem

$$\begin{array}{ccc} \text{data} & \xrightarrow{\text{identification}} & \text{model} \\ w & & \mathscr{B} \in \mathscr{M} \end{array}$$

the aim is to obtain "simple" and "accurate" model:

"accurate" $\rightarrow$ min. error$(w, \widehat{\mathscr{B}}) =$ misfit/latency

"simple" $\rightarrow$ Occam's razor principle:
among equally accurate models,
choose the simplest

Model complexity

simple models are small models

$$\mathscr{B}_1 \subset \mathscr{B}_2 \quad \Longrightarrow \quad \mathscr{B}_1 \text{ is simpler than } \mathscr{B}_2$$

nonlinear model complexity is an open problem

in the linear time-invariant case, $\mathscr{B}$ is a subspace

$$\text{size of the model} = \text{dimension of } \mathscr{B}$$

however, models with inputs are infinite dimensional

Linear time-invariant model's complexity

restriction of $\mathscr{B}$ on an interval $[1, T]$

$$\mathscr{B}|_T = \{\, w = \big(w(1), \dots, w(T)\big) \mid \exists\, w_{\text{p}}, w_{\text{f}}, \text{ such that } (w_{\text{p}}, w, w_{\text{f}}) \in \mathscr{B} \,\}$$

for sufficiently large T

$$\dim(\mathscr{B}|_T) = (\#\text{ of inputs}) \cdot T + (\text{order})$$

$$\text{complexity}(\mathscr{B}) = \begin{bmatrix} \mathtt{m} \\ \ell \end{bmatrix} \begin{array}{l} \rightarrow \ \#\text{ of inputs} \\ \rightarrow \ \text{order or lag} \end{array}$$

$\mathscr{L}_{\mathtt{m},\ell}$ — set of LTI systems of bounded complexity

Complexity selection

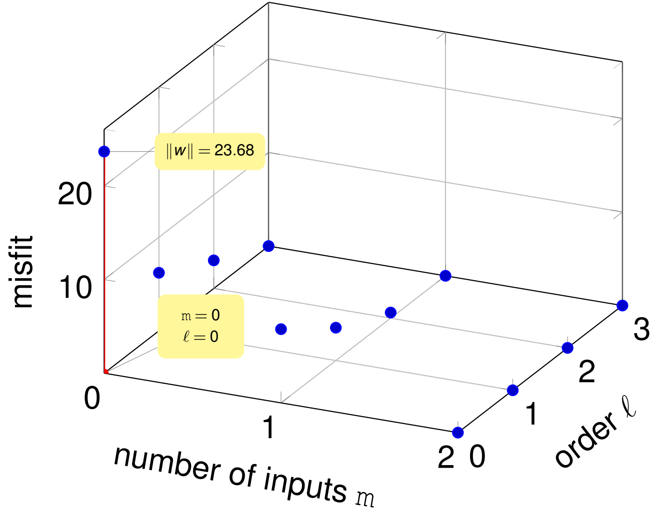
if `m` is given and fixed, choosing the complexity is an

order selection problem

in general, choosing the complexity involves

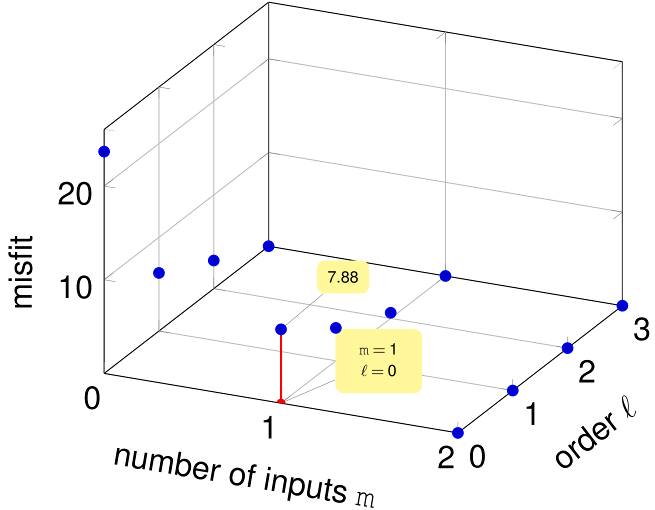
order selection and input selection

Example: misfit-complexity trade-off



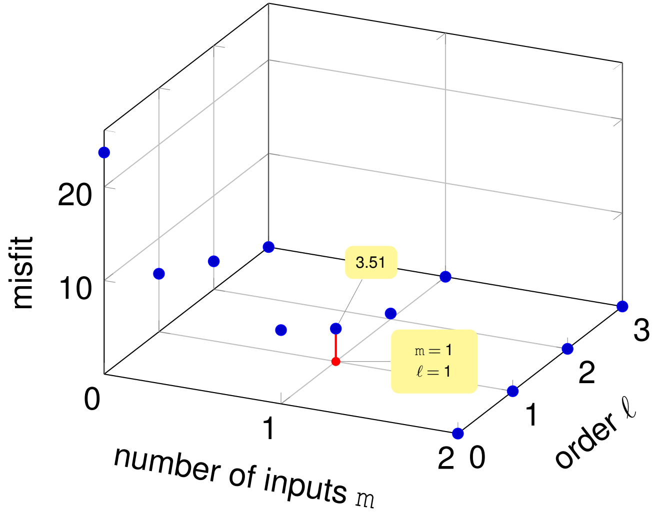
$\mathtt{m} = 0$, $\ell = 0 \implies \mathscr{B} = \{0\}$ is the only model

Example: misfit-complexity trade-off



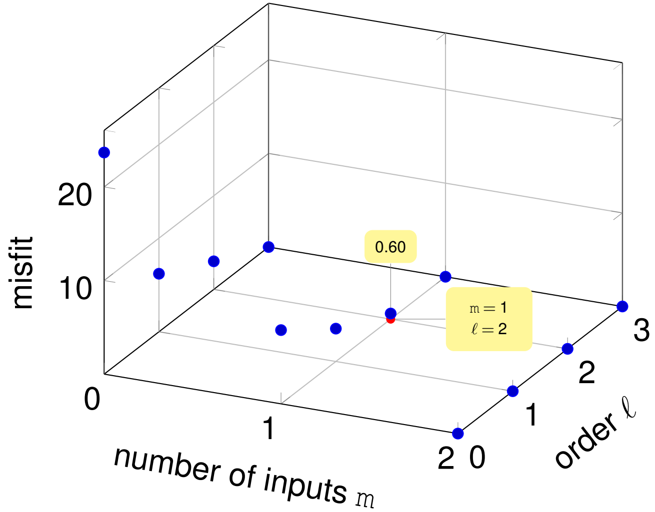
$m = 1$, $\ell = 0 \implies \mathscr{B}$ is a line through 0

Example: misfit-complexity trade-off



$\mathtt{m} = 1$, $\ell = 1 \implies \mathscr{B}$ is 1st order SISO

Example: misfit-complexity trade-off



$\mathtt{m}=1$, $\ell=2$ $\implies$ $\mathscr{B}$ is 2nd order SISO

Approximation error-complexity trade-off

$$\text{minimize} \quad \text{over } \widehat{\mathscr{B}} \in \mathscr{L} \quad \begin{bmatrix} \text{error}(w, \widehat{\mathscr{B}}) \\ \text{complexity}(\widehat{\mathscr{B}}) \end{bmatrix}$$

three ways to "scalarize" the problem:

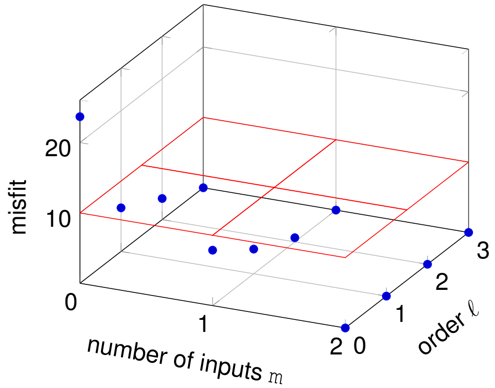
1. minimize over $\widehat{\mathscr{B}} \in \mathscr{L}$ $\text{error}(w, \widehat{\mathscr{B}}) + \lambda\,\text{complexity}(\widehat{\mathscr{B}})$

2. $$\begin{array}{ll} \text{minimize} & \text{over } \widehat{\mathscr{B}} \in \mathscr{L} \quad \text{complexity}(\widehat{\mathscr{B}}) \\ \text{subject to} & \text{error}(w, \widehat{\mathscr{B}}) \leq \mu \end{array}$$

3. $$\begin{array}{ll} \text{minimize} & \text{over } \widehat{\mathscr{B}} \quad \text{error}(w, \widehat{\mathscr{B}}) \\ \text{subject to} & \widehat{\mathscr{B}} \in \mathscr{L}_{\mathtt{m},\ell} \end{array}$$

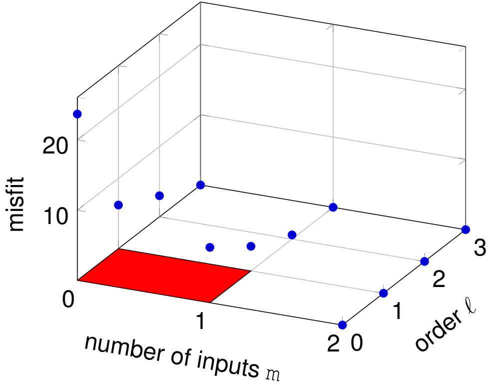
Complexity minimization with error bound

$$\begin{array}{ll} \text{minimize} & \text{over } \widehat{\mathscr{B}} \in \mathscr{L} \quad \text{complexity}(\widehat{\mathscr{B}}) \\ \text{subject to} & \text{error}(w, \widehat{\mathscr{B}}) \leq \mu \end{array}$$



Error minimization with complexity bound

$$\begin{array}{ll} \text{minimize} & \text{over } \widehat{\mathscr{B}} \quad \text{error}(w, \widehat{\mathscr{B}}) \\ \text{subject to} & \widehat{\mathscr{B}} \in \mathscr{L}_{\mathtt{m},\ell} \end{array}$$



Summary: error-complexity trade-off

LTI model complexity

$$\text{complexity}(\mathscr{B}) = \begin{bmatrix} \mathtt{m} \\ \ell \end{bmatrix} \begin{matrix} \to & \text{\# of inputs} \\ \to & \text{order or lag} \end{matrix}$$

error-complexity trade-off

$$\text{minimize} \quad \text{over } \widehat{\mathscr{B}} \in \mathscr{L} \quad \begin{bmatrix} \text{error}(w, \widehat{\mathscr{B}}) \\ \text{complexity}(\widehat{\mathscr{B}}) \end{bmatrix}$$

tracing all optimal solutions requires hyper parameter

1. λ — no physical meaning
2. μ — bound on the error
3. $(\mathtt{m}, \ell)$ — bound on the complexity

Outline

Approximation error–model complexity trade-off

System identification $\leftrightarrow$ low-rank approximation

Solution methods: variable projection

Approximate identification problem

$$\text{minimize} \quad \text{over } \widehat{\mathscr{B}} \quad \text{error}(w, \widehat{\mathscr{B}})$$
$$\text{subject to} \quad \widehat{\mathscr{B}} \in \mathscr{L}_{\mathrm{m},\ell}$$

in the case error $=$ misfit

$$\text{error}(w, \widehat{\mathscr{B}}) = \min_{\widehat{w} \in \widehat{\mathscr{B}}} \|w - \widehat{w}\|$$

the problem is

$$\text{minimize} \quad \text{over } \widehat{\mathscr{B}}, \widehat{w} \quad \|w - \widehat{w}\|$$
$$\text{subject to} \quad \widehat{w} \in \widehat{\mathscr{B}} \in \mathscr{L}_{\mathrm{m},\ell}$$

Exact, noisy, and missing data

$v_i^k(t)$ — variance of the measurement noise on $w_i^k(t)$

$$\|w - \widehat{w}\|_\alpha^2 = \sum_{k=1}^{N} \sum_{i=1}^{q} \sum_{t=1}^{T} \alpha_i^k(t) \big(w_i^k(t) - \widehat{w}_i^k(t) \big)^2$$

noisy data

$$\alpha_i^k(t) := \frac{1}{v_i^k(t)}$$

exact data

$v_i^k(t) = 0$, $\alpha_i^k(t) = \infty$

missing data

$v_i^k(t) = \infty$, $\alpha_i^k(t) = 0$

$v_i^k(t) = \infty$ imposes equality constraint $\widehat{w}_i^k(t) = w_i^k(t)$

$v_i^k(t) = 0$ makes $\|w - \widehat{w}\|_\alpha^2$ independent of $w_i^k(t)$

Summary: identification problem

approximate identification in the misfit setting

$$\begin{array}{lll} \text{minimize} & \text{over } \widehat{\mathscr{B}}, \widehat{w} & \|w - \widehat{w}\|_\alpha \\ \text{subject to} & \widehat{w} \in \widehat{\mathscr{B}} \in \mathscr{L}_{\mathtt{m},\ell} & \end{array} \qquad \text{(SYSID)}$$

element-wise weighted error criterion $\|\cdot\|_\alpha$

$$\begin{array}{llll} \text{exact} & w_i^k(t) & \leftrightarrow & \alpha_i^k(t) = \infty \\ \text{missing} & w_i^k(t) & \leftrightarrow & \alpha_i^k(t) = 0 \end{array}$$

Next: SYSID $\leftrightarrow$ Hankel structured LRA

exact trajectory $w \in \mathscr{B} \in \mathscr{L}_{\mathrm{m},\ell}$

$$\Updownarrow$$

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_\ell w(t+\ell) = 0$$

$$\Updownarrow$$

rank deficient

$$\mathscr{H}_{\ell+1}(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-\ell) \\ w(2) & w(3) & \cdots & w(T-\ell+1) \\ w(3) & w(4) & \cdots & w(T-\ell+2) \\ \vdots & \vdots & & \vdots \\ w(\ell+1) & w(\ell+2) & \cdots & w(T) \end{bmatrix}$$

$w \in \mathscr{B} \iff \mathscr{H}_{\ell+1}(w)$ rank deficient

relation at time $t = 1$

$$R_0 w(1) + R_1 w(2) + \cdots + R_\ell w(\ell+1) = 0$$

in matrix form:

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_\ell \end{bmatrix} \begin{bmatrix} w(1) \\ w(2) \\ \vdots \\ w(\ell+1) \end{bmatrix} = 0$$

$w \in \mathscr{B} \iff \mathscr{H}_{\ell+1}(w)$ rank deficient

relation at time $t = 2$

$$R_0 w(2) + R_1 w(3) + \cdots + R_\ell w(\ell+2) = 0$$

in matrix form:

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_\ell \end{bmatrix} \begin{bmatrix} w(2) \\ w(3) \\ \vdots \\ w(\ell+2) \end{bmatrix} = 0$$

$w \in \mathscr{B} \iff \mathscr{H}_{\ell+1}(w)$ rank deficient

relation at time $t = T - \ell$

$$R_0 w(T-\ell) + R_1 w(T-\ell+1) + \cdots + R_\ell w(T) = 0$$

in matrix form:

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_\ell \end{bmatrix} \begin{bmatrix} w(T-\ell) \\ w(T-\ell+1) \\ w(T-\ell+2) \\ \vdots \\ w(T) \end{bmatrix} = 0$$

Putting it all together

relation for $t = 1, \ldots, T - \ell$

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_\ell w(t+\ell) = 0$$

in matrix form:

$$\underbrace{\begin{bmatrix} R_0 & R_1 & \cdots & R_\ell \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} w(1) & w(2) & \cdots & w(T-\ell) \\ w(2) & w(3) & \cdots & w(T-\ell+1) \\ w(3) & w(4) & \cdots & w(T-\ell+2) \\ \vdots & \vdots & & \vdots \\ w(\ell+1) & w(\ell+2) & \cdots & w(T) \end{bmatrix}}_{\mathscr{H}_{\ell+1}(w)} = 0$$

$w \in \mathscr{B} \iff \mathscr{H}_{\ell+1}(w)$ rank deficient

with $R \in \mathbb{R}^{(q-\mathtt{m}) \times q(\ell+1)}$ full row rank,

$$\operatorname{rank}\big(\mathscr{H}_{\ell+1}(w) = 0\big) \leq q\ell + \mathtt{m} \qquad (q \text{ — \# of variables})$$

$$w \in \mathscr{B} \in \mathscr{L}_{\mathtt{m},\ell} \iff \operatorname{rank}\big(\mathscr{H}_{\ell+1}(w)\big) \leq q\ell + \mathtt{m}$$

multiple time-series $\quad \leftrightarrow \quad$ mosaic-Hankel matrix

$$\{w^1, \dots, w^N\} \subset \mathscr{B} \in \mathscr{L}_{\mathtt{m},\ell}$$

$$\iff \operatorname{rank}\Big(\underbrace{\begin{bmatrix} \mathscr{H}_{\ell+1}(w^1) & \cdots & \mathscr{H}_{\ell+1}(w^N) \end{bmatrix}}_{\mathscr{H}_{\ell+1}(w)}\Big) \leq q\ell + \mathtt{m}$$

Structured weighted low-rank approximation

$$\begin{array}{lll} \text{minimize} & \text{over } \widehat{\mathscr{B}} \text{ and } \widehat{w} & \|w - \widehat{w}\|_\alpha \\ \text{subject to} & \widehat{w} \subset \widehat{\mathscr{B}} \in \mathscr{L}_{\mathrm{m},\ell} & \end{array} \qquad \text{(SYSID)}$$

$$\Updownarrow$$

$$\begin{array}{lll} \text{minimize} & \text{over } \widehat{w} & \|w - \widehat{w}\|_\alpha \\ \text{subject to} & \operatorname{rank}\big(\mathscr{H}_{\ell+1}(\widehat{w})\big) \leq q\ell + \mathrm{m} & \end{array} \qquad \text{(SLRA)}$$

Summary: structured low-rank approximation

(SYSID) $\iff$ (SLRA)

LTI model class $\iff$ Hankel structure

repeated experiments $\iff$ mosaic-Hankel structure

$$\begin{bmatrix} \mathscr{H}_{\ell+1}(w^1) & \cdots & \mathscr{H}_{\ell+1}(w^N) \end{bmatrix}$$

bounded complexity $\iff$ rank constraint

$$(\mathtt{m}, \ell) \quad \leftrightarrow \quad r = q\ell + \mathtt{m}$$

Outline

Approximation error–model complexity trade-off

System identification $\leftrightarrow$ low-rank approximation

Solution methods: variable projection

Solution methods

given: data w and complexity bound $(\mathtt{m}, \ell)$

find: $\widehat{\mathscr{B}}$ that solves (SYSID) or, equivalently, (SLRA)

1. choice of model representation
    - transfer function
    - input/state/output
    - ...
2. choice of optimization method
    - local optimization
    - global optimization
    - convex relaxations

Model vs model representation

1st order SISO model $\mathscr{B} \in \mathscr{L}_{1,1}$

$$\mathscr{B}_{\mathrm{de}}(\theta) = \left\{ \widehat{w} \;\middle|\; \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix} \begin{bmatrix} \widehat{w}_1(t) \\ \widehat{w}_2(t) \\ \widehat{w}_1(t+1) \\ \widehat{w}_2(t+1) \end{bmatrix} = 0, \ \forall t \right\}$$

transfer functions

$$G_{w_1 \mapsto w_2}(z) = -\frac{\theta_1 + \theta_3 z}{\theta_2 + \theta_4 z} \quad , \quad G_{w_2 \mapsto w_1}(z) = -\frac{\theta_2 + \theta_4 z}{\theta_1 + \theta_3 z}$$

state space, convolution, ..., representations

Problem formulation vs solution method

in the classical setting, model $=$ representation

$\implies$ problems are mixed with solution methods

e.g., "total least-squares" is both problem and method

the behavioral setting distinguishes

	used for	involves
abstract	problem formulation	$\mathscr{B}$, $\mathscr{L}_{\mathrm{m},\ell}$
concrete	solution methods	$\mathscr{B}(\theta)$, $\theta \in \Theta$

low-rank approx. is abstract problem formulation

Parameter optimization problem

model representation

$$\mathscr{B}(\theta) = \{\, \widehat{w} \mid g_\theta(\widehat{w}) = 0 \,\}$$

parameterized model class

$$\mathscr{M} = \{\, \mathscr{B}(\theta) \mid \theta \in \Theta \,\}$$

optimization problem

$$\begin{array}{lll} \text{minimize} & \text{over } \theta \in \Theta,\ \widehat{w} & \|w - \widehat{w}\|_\alpha \\ \text{subject to} & g_\theta(\widehat{w}) = 0 & \end{array} \qquad (\mathrm{SYSID}_\theta)$$

Bilinear structure of the problem

(SYSID_θ) — constrained nonlinear least-squares

$\mathscr{B}$ linear

$\implies \quad g_\theta(\widehat{w})$ bilinear (in θ and $\widehat{w}$)

$\implies \quad$ (SYSID_θ) can be solved globally for given θ

variable projection (VARPRO)
for separable nonlinear least-squares problems

if $T \gg \ell$, elimination of $\widehat{w}$ leads to big reduction

System theoretic view of VARPRO

solving (SYSID_θ) for given θ

$$\Updownarrow$$

misfit evaluation: $\text{error}\big(w, \mathscr{B}(\theta)\big)$

$$\Updownarrow$$

likelihood evaluation

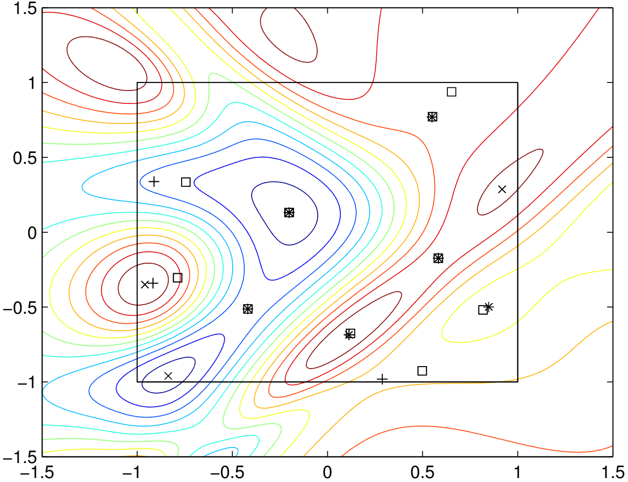
$$\Updownarrow$$

least-squares smoothing of w by $\mathscr{B}(\theta)$

$$\Updownarrow$$

fast algorithms:
Kalman smoothing
Cholesky factorization
...

Non-convexity of error$(w, \mathscr{B}(\theta))$



Computational details

$O(T)$ evaluation of $\text{error}\big(w, \mathscr{B}(\theta)\big)$ and its derivatives

- using the Kalman smoother
- Cholesky factorization of banded Toeplitz matrix
- ...

$\mathscr{B}(\theta) = \mathscr{B}(\alpha\theta)$, for all $\alpha \neq 0$

$\Theta = \{\, \theta \mid \|\theta\|_2 = 1 \,\} \implies$ optimization on a manifold

- generic methods (optimization theory)
- custom methods (system identification)
  - data driven local coordinates (McKelvey)
  - ...

Summary: solution methods

solution methods involve two choices:

1. model representation
2. optimization method

in the linear case, bilinear structure $\rightsquigarrow$ VARPRO

constraint nonlinear least-squares problem

$$\min_{\theta \in \Theta} \; \text{error}\big(w, \mathscr{B}(\theta)\big)$$

Θ is a manifold $\rightsquigarrow$ optimization on a manifold