DYSCO course on low-rank approximation and its applications

Homework

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1 Total least squares

1.1 Unconstrained problem, equivalent to total least squares

Assignment **PROB**

A total least squares approximate solution x_{tls} of the linear system of equations $Ax \approx b$ is a solution to the following optimization problem

$$\text{minimize} \quad \text{over } x, \widehat{A}, \text{ and } \widehat{b} \quad \left\| \begin{bmatrix} A & b \end{bmatrix} - \begin{bmatrix} \widehat{A} & \widehat{b} \end{bmatrix} \right\|_{\text{F}}^2 \quad \text{subject to} \quad \widehat{A}x = \widehat{b}. \tag{TLS}$$

Show that (TLS) is equivalent to the unconstrained optimization problem

$$\text{minimize } f_{\text{tls}}(x), \quad \text{where} \quad f_{\text{tls}}(x) := \frac{\|Ax - b\|_2^2}{\|x\|_2^2 + 1}. \tag{TLS'}$$

Give an interpretation of the function f_{tls}.

1.2 Lack of total least squares solution

Assignment **PROB**

Using the formulation (TLS'), derived in Problem 1.1, show that the total least squares line fitting problem

$$\begin{aligned} &\text{minimize} \quad \text{over } x \in \mathbb{R}, \widehat{a} \in \mathbb{R}^N, \text{ and } \widehat{b} \in \mathbb{R}^N \quad \sum_{j=1}^{N} \left\| d_j - \begin{bmatrix} \widehat{a}_j \\ \widehat{b}_j \end{bmatrix} \right\|_2^2 \\ &\text{subject to} \quad \widehat{a}_j x = \widehat{b}_j, \quad \text{for } j = 1, \dots, N \end{aligned} \tag{tls}$$

has no solution for the data

$$\begin{aligned} d_1 &= \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad d_3 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \quad d_4 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad d_5 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \\ d_6 &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad d_7 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad d_8 = \begin{bmatrix} 0 \\ -6 \end{bmatrix}, \quad d_9 = \begin{bmatrix} -1 \\ -4 \end{bmatrix}, \quad d_{10} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}. \end{aligned} \tag{data}$$

1.3 Quadratically constrained problem, equivalent to rank-1 approximation

Assignment **PROB**

Show that

$$\begin{aligned} &\text{minimize} \quad \text{over } P \in \mathbb{R}^{2\times 1} \text{ and } L \in \mathbb{R}^{1\times N} \quad \|D - \widehat{D}\|_{\text{F}}^2 \\ &\text{subject to} \quad \widehat{D} = PL. \end{aligned} \tag{lra$_P$}$$

is equivalent to the quadratically constrained optimization problem

$$\text{minimize } f_{\text{lra}}(P) \quad \text{subject to} \quad P^\top P = 1, \tag{lra$'_P$}$$

where

$$f_{\text{lra}}(P) = \text{trace}\big(D^\top (I - PP^\top)D\big).$$

Explain how to find all solutions of (lra_P) from a solution of (lra'_P). Assuming that a solution to (lra'_P) exists, is it unique?

1.4 Analytic solution of a rank-1 approximation problem

Assignment **PROB**

Show that for the data (data),

$$f_{\text{lra}}(P) = P^\top \begin{bmatrix} 140 & 0 \\ 0 & 20 \end{bmatrix} P.$$

Using geometric or analytic arguments, conclude that the minimum of f_{lra} for a P on the unit circle is 20 and is achieved for

$$P^{*,1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad P^{*,2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{$*$}$$

1.5 Analytic solution of two-variate rank-1 approximation problem

Assignment **PROB**

Find an analytic solution of the Frobenius norm rank-1 approximation of a $2 \times N$ matrix.

1.6 Analytic solution of scalar total least squares

Assignment **PROB**

Find an analytic expression for the total least squares solution of the system $ax \approx b$, where $a, b \in \mathbb{R}^m$.

2 Exact identification

2.1 State space identification of an LTI autonomous model

Assignment **PROB**

Given a trajectory $y = \big(y(1), y(2), \dots, y(T)\big)$ of an autonomous linear time-invariant system $\mathscr{B}$ of order $\mathtt{n}$, find a state space representation $\mathscr{B}_{\text{i/s/o}}(A, C)$ of $\mathscr{B}$. Modify your procedure, so that it does not require prior knowledge of the system order $\mathtt{n}$ but only an upper bound $\mathtt{n}_{\max}$ for it.

2.2 Identification of of a general LTI model

Assignment **PROB**

In Lecture 5, we outlined the following algorithms for exact system identification:

1. $w_{\text{d}} \mapsto R(z)$, where $\widehat{\mathscr{B}} := \ker\big(R(z)\big)$ is the identified model,
2. $w_{\text{d}} \mapsto H$, where H contains the first samples of the impulse response of~$\widehat{\mathscr{B}}$,
3. $w_{\text{d}} \mapsto \mathscr{O}_{\ell_{\max}+1}(A, C) \mapsto (A, B, C, D)$, where (A, B, C, D) is an input/state/output representation of~$\widehat{\mathscr{B}}$, and
4. $w_{\text{d}} \mapsto \big(x_{\text{d}}(1), \dots, x_{\text{d}}(\mathtt{n}_{\max} + \mathtt{m} + 1)\big) \mapsto (A, B, C, D)$.

Implement algorithms 1 and 4 and apply them on the data available from

`http://homepages.vub.ac.be/~imarkovs/dysco/exactid_data.mat`

The computed model parameters $\widehat{R}(z)$ and $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$ define models $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$. Verify that the models $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ are exact for w_d, *i.e.*, $w_\text{d} \in \widehat{\mathscr{B}}_1$ and $w_\text{d} \in \widehat{\mathscr{B}}_2$, and are equivalent, *i.e.*, $\widehat{\mathscr{B}}_1 = \widehat{\mathscr{B}}_2$.

3 Approximate identification

3.1 A simple method for approximate system identification

Assignment **PROB**

Modify the algorithms developed in Section 2, so that they can be used as approximate identification methods. (You can assume that the system is single input single output and the order is known.)

3.2 Algorithms for approximate system identification

Assignment **PROB**

1. Download the file `flutter.dat` from DAISY.
2. Download and install the `slra` package.
3. Partition the data set into *identification* (*e.g.*, first 60%) and *validation* (*e.g.*, remaining 40%) parts.
4. Apply the functions developed in Problem 3.1 on the identification part of the data. In this and all steps below use model order $\mathtt{n} = 3$.
5. Apply the approximate identification method `ident` from the `slra` package on the identification part of the data.
6. A classical method for system identification is the prediction error method (PEM) and a popular implementation of the PEM method is the function `pem` from the System Identification Toolbox of Matlab. Similarly to the misfit minimization methods, PEM is based on local optimization, starting from an initial approximation. Apply the function `pem` on the identification part of the data.
7. A validation function from the System Identification Toolbox is `compare`. Using the functions `compare` and `misfit`, validate the models identified by all methods.
8. Repeat steps 3–5 for different partitions of the data into identification and validation parts. Comment on the results.