Realization and identification of autonomous linear periodically time-varying systems

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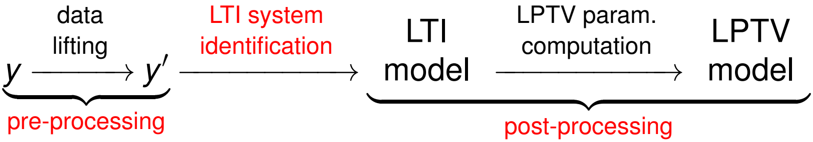
Outline



1. $O(T\mathrm{p}L^2)$ algorithm for LPTV system realization
    - T — # of samples
    - p — # of outputs ($\dim(y)$)
    - L — upper bound of the order
2. algorithm for LPTV maximum likelihood identification with $O(T\mathrm{p}L^2)$ cost per iteration

Autonomous LPTV systems

- state space representation $\qquad$ (σ — shift operator)

$$\mathscr{B}(A,C) := \{\, y \mid \sigma x = Ax,\ y = Cx,\ x(1) = x_{\text{ini}} \in \mathbb{R}^{n} \,\}$$

- change of basis, *i.e.*,

$$\mathscr{B} = \mathscr{B}(A,C) = \mathscr{B}(\widehat{A},\widehat{C})$$

$$\widehat{A} = \sigma V A V^{-1}, \qquad \widehat{C} = C V^{-1}$$

- *P*-periodicity

$$A = \sigma^{P} A, \qquad C = \sigma^{P} C, \qquad V = \sigma^{P} V$$

Problem formulation

realization

- given: $y = \big(y(1), \dots, y(T)\big)$, period P, and order $\mathtt{n}$
- find: $\widehat{A}$, $\widehat{C}$, such that $y \in \mathscr{B}(\widehat{A}, \widehat{C})$

identification

- given: $y = \big(y(1), \dots, y(T)\big)$, period P, and order $\mathtt{n}$

$$\begin{array}{ll} \text{minimize} & \text{over } \widehat{y} \text{ and } \widehat{\mathscr{B}} \quad \|y - \widehat{y}\|_2 \\ \text{subject to} & \widehat{y} \in \widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P} \end{array}$$

$\mathscr{L}_{0,\mathtt{n},P}$ — set of autonomous LPTV systems with order at most $\mathtt{n}$ and period P $(0 =$ no inputs)

“lifting” operator

$$\big(y(1),\dots,y(T)\big) = y \mapsto y' = \big(y'(1),\dots,y'(T')\big),\ T' := \lfloor T/P \rfloor$$

$$y' = \mathrm{lift}_P(y) = \left(\begin{bmatrix} y(1) \\ \vdots \\ y(P) \end{bmatrix}, \begin{bmatrix} y(P+1) \\ \vdots \\ y(2P) \end{bmatrix}, \dots, \begin{bmatrix} y((T'-1)P) \\ \vdots \\ y(T'P) \end{bmatrix} \right)$$

Theorem 1

- $\mathscr{B}(A,C)$ — LPTV of order $\mathtt{n}$, period P, with $\mathtt{p}$ outputs
- $\mathrm{lift}_P\big(\mathscr{B}(A,C)\big)$ — LTI of order $\mathtt{n}$, with $\mathtt{p}' := \mathtt{p}P$ outputs

Identification of the lifted system

- $\text{lift}_P\big(\mathscr{B}(A,C)\big)$ admits $\mathtt{n}$th order repr. $\mathscr{B}(\Phi,\Psi)$
- $y' \mapsto (\widehat{\Phi},\widehat{\Psi})$ is classical realization problem
- can be solved, *e.g.*, by Kung's method

$$\underbrace{\mathscr{H}_L(y')}_{\text{Hankel}} = \underbrace{\mathscr{O}(\widehat{\Phi},\widehat{\Psi})}_{\mathbf{O}}\underbrace{\mathscr{O}(\widehat{\Phi}^\top,\widehat{x}_{\text{ini}}^\top)}_{\mathbf{C}}$$

$$\mathbf{O} \in \mathbb{R}^{L\mathrm{p}'\times\mathrm{n}} \qquad \mathbf{C} \in \mathbb{R}^{\mathrm{n}\times(T'-L)}$$

- $\widehat{\Phi}^\top$ is a solution of the shift equation

$$\underline{\mathbf{O}}\widehat{\Phi} = \overline{\mathbf{O}}$$

- $\widehat{\Psi}$ is the first block-element of $\mathbf{O}$

Computation of the model parameters

Theorem 2

- define $\widehat{A} = \big(\widehat{A}_1, \ldots, \widehat{A}_P\big)$ and $\widehat{C} = \big(\widehat{A}_1, \ldots, \widehat{A}_P\big)$ via

$$\widehat{A}_1 = \ldots = \widehat{A}_{P-1} = I_n, \quad \widehat{A}_P := \widehat{\Phi}$$

$$\operatorname{col}(\widehat{C}_1, \ldots, \widehat{C}_P) := \widehat{\Psi}, \quad \widehat{C}_i \in \mathbb{R}^{\mathrm{p} \times \mathrm{n}}$$

(note that $\widehat{\Psi} = \widehat{A}_P \widehat{A}_{P-1} \cdots \widehat{A}_2 \widehat{A}_1$)

- $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$ (LTI) is equivalent to $\mathscr{B}(\widehat{A}, \widehat{C})$ (LPTV), *i.e.*,

$$\mathscr{B}(\widehat{\Phi}, \widehat{\Psi}) = \operatorname{lift}_P\big(\mathscr{B}(\widehat{A}, \widehat{C})\big)$$

Proof

- we have to show that a response y of $\text{lift}_P\big(\mathscr{B}(\widehat{A}, \widehat{C})\big)$ is also a response of $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$ and vice verse

- the response of $\mathscr{B}(\widehat{A}, \widehat{C})$ to x_{ini} is

$$\begin{gathered} y(1) = \widehat{C}_1 x_{\text{ini}}, \quad \dots, \quad y(P) = \widehat{C}_P x_{\text{ini}} \\ y(P+1) = \widehat{C}_1 \widehat{\Phi} x_{\text{ini}}, \quad \dots, \quad y(2P) = \widehat{C}_P \widehat{\Phi} x_{\text{ini}} \\ \vdots \\ y(t'P+1) = \widehat{C}_1 \widehat{\Phi}^{t'} x_{\text{ini}}, \quad \dots, \quad y\big((t'+1)P\big) = \widehat{C}_P \widehat{\Phi}^{t'} x_{\text{ini}} \end{gathered}$$

- the response of $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$ to x_{ini} is

$$y'(1) = \widehat{\Psi} x_{\text{ini}}, \quad y'(2) = \widehat{\Psi}\widehat{\Phi} x_{\text{ini}}, \quad \dots, \quad y'(t') = \widehat{\Psi}\widehat{\Phi}^{t'} x_{\text{ini}}$$

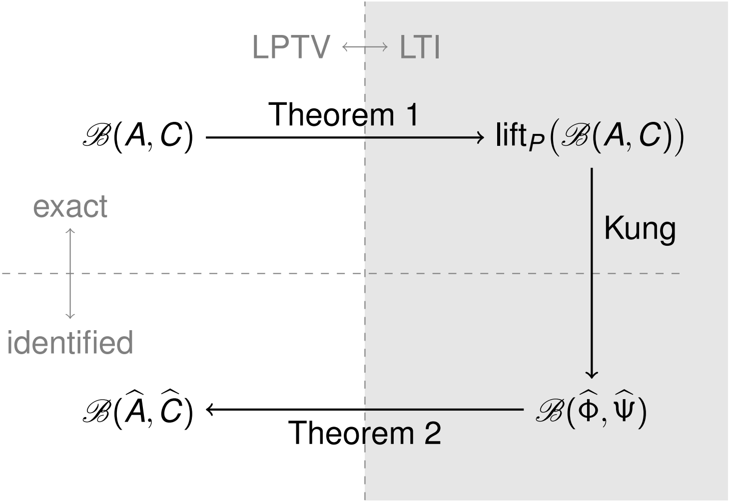
- by the definition of Ψ, we have

$$y'(1) = \begin{bmatrix} \widehat{C}_1 \\ \vdots \\ \widehat{C}_P \end{bmatrix} x_{\text{ini}}, \quad \dots, \quad y'(t') = \begin{bmatrix} \widehat{C}_1 \\ \vdots \\ \widehat{C}_P \end{bmatrix} \widehat{\Phi}^{t'} x_{\text{ini}}$$

- this shows that the two responses are equivalent up to the reshaping done by lift_P

$$\begin{array}{ccc} \mathscr{B}(A,C) & \xrightarrow{\text{Theorem 1}} & \mathrm{lift}_P\big(\mathscr{B}(A,C)\big) \\ & & \Big\downarrow \text{Kung} \\ \mathscr{B}(\widehat{A},\widehat{C}) & \xleftarrow[\text{Theorem 2}]{} & \mathscr{B}(\widehat{\Phi},\widehat{\Psi}) \end{array}$$

Summary


LPTV ⟷ LTI
Theorem 1
$\mathscr{B}(A, C)$
$\mathrm{lift}_P\big(\mathscr{B}(A, C)\big)$
Kung
exact
identified
$\mathscr{B}(\widehat{A}, \widehat{C})$
$\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$
Theorem 2

Modified method

- using the “transposed” lifted sequence

$$y'^{\top} := \left(y'^{\top}(1), \dots, y'^{\top}(T')\right), \qquad y'^{\top}(t) \in \mathbb{R}^{1\times \mathrm{p}'}$$

- the Hankel matrix factorization becomes

$$\underbrace{\mathscr{H}_L(y'^{\top})}_{\text{Hankel}} = \underbrace{\mathscr{O}_L(\widehat{\Phi}^{\top}, x_{\text{ini}}^{\top})}_{\mathbf{O}} \cdot \underbrace{\mathscr{O}_{T'-L+1}^{\top}(\widehat{\Phi}, \widehat{\Psi})}_{\mathbf{C}} \qquad \begin{array}{l} \mathbf{O} \in \mathbb{R}^{L\times \mathrm{n}} \\ \mathbf{C} \in \mathbb{R}^{\mathrm{n}\times \mathrm{p}(T-L)} \end{array}$$

- $\widehat{\Phi}^{\top}$ is a solution of the shift equation

$$\underline{\mathbf{O}}\widehat{\Phi}^{\top} = \overline{\mathbf{O}}$$

- $\widehat{\Psi}^{\top}$ is the first block element of $\mathbf{C}$

Identification

- the SYSID problem is equivalent to

$$\begin{array}{lll} \text{minimize} & \text{over } \widehat{y} & \|y-\widehat{y}\|_2 \\ \text{subject to} & \operatorname{rank}\big(\mathscr{H}_{\mathrm{n}+1}\big(\operatorname{lift}_P(\widehat{y}^\top)\big)\big) \leq \mathrm{n} & \end{array} \qquad \text{(SLRA)}$$

- (SLRA) is Hankel structured low-rank approximation

- existing methods can be used, e.g.,

$$\operatorname{rank}\big(\mathscr{H}_{\mathrm{n}+1}\big(\operatorname{lift}_P(\widehat{y}^\top)\big)\big) \leq \mathrm{n} \quad \iff \quad \exists\ R^{1\times(\mathrm{n}+1)}, \quad R\mathscr{H}_{\mathrm{n}+1}\big(\operatorname{lift}_P(\widehat{y}^\top)\big) = 0, \quad RR^\top = 1$$

Simulation setup

- the data is generated by an output error model

  $$y = \bar{y} + \widetilde{y}, \quad \text{where} \quad \bar{y} \in \bar{\mathscr{B}} \in \mathscr{L}_{0,\mathrm{n},P} \text{ and } \widetilde{y} \sim \mathrm{N}(0, s^2 I_{\mathrm{p}})$$

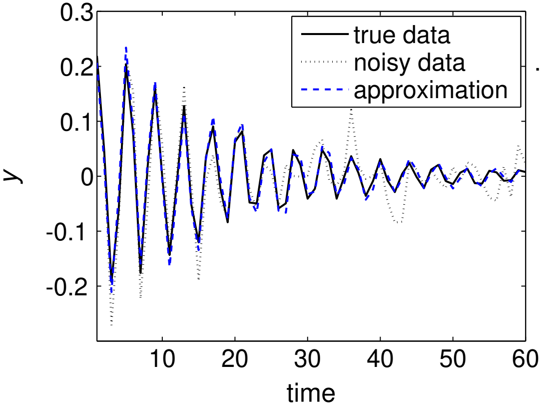
  split into identification (3/4) and validation (1/4) parts

- $\bar{\mathscr{B}}$ is Mathieu oscillator—spring-mass-damper system with time-periodic spring stiffness

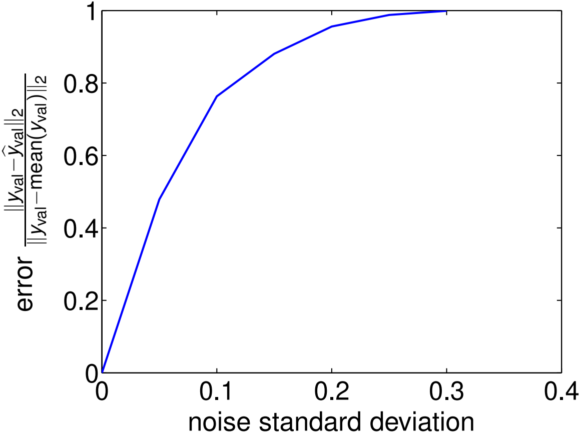
  $$\bar{A}_\tau = \begin{bmatrix} 0 & 1 \\ \bar{a}_1 & \bar{a}_{2,\tau} \end{bmatrix}, \quad \bar{C}_\tau = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

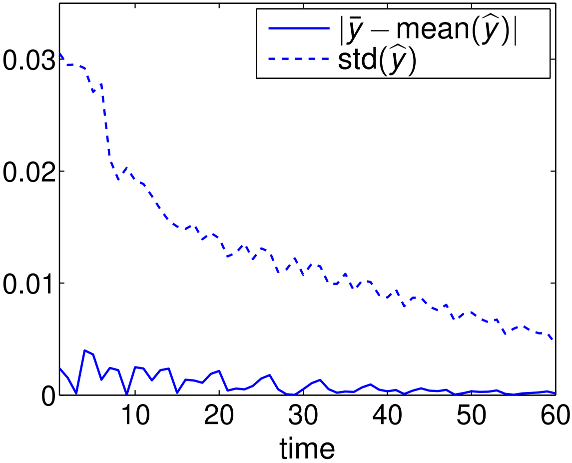
- in the example $P = 3$ and $T' = 20$

The data and its approximation



Average approximation error




$|\bar{y} - \mathrm{mean}(\widehat{y})|$
$\mathrm{std}(\widehat{y})$
0.03
0.02
0.01
0
10
20
30
40
50
60
time

Definitions

- block-Hankel matrix

$$\mathscr{H}_L(y) := \begin{bmatrix} y(1) & y(2) & y(3) & \cdots & y(T-L+1) \\ y(2) & y(3) & ⋰ & & y(T-L+1) \\ y(3) & ⋰ & & & \vdots \\ \vdots & & & & \\ y(L) & y(L+1) & \cdots & & y(T) \end{bmatrix}$$

- extended observability matrix

$$\mathscr{O}_L(A,C) := \begin{bmatrix} C(1) \\ C(2)A(1) \\ C(3)A(2)A(1) \\ \vdots \\ C(L)A(L-1)A(L-2)\cdots A(1) \end{bmatrix}$$

References

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