Nonlinear state-space modeling

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Linear state-space model

Continuous-time

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

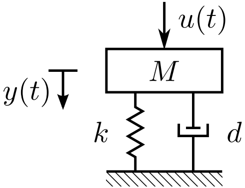
Discrete-time

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$$\begin{aligned} x(t) &\in \mathbb{R}^{n} \\ u(t) &\in \mathbb{R}^{n_u} \\ y(t) &\in \mathbb{R}^{n_y} \end{aligned}$$

Example: mass-spring-damper

$$M\ddot{y}(t) + d\dot{y}(t) + ky(t) = u(t)$$



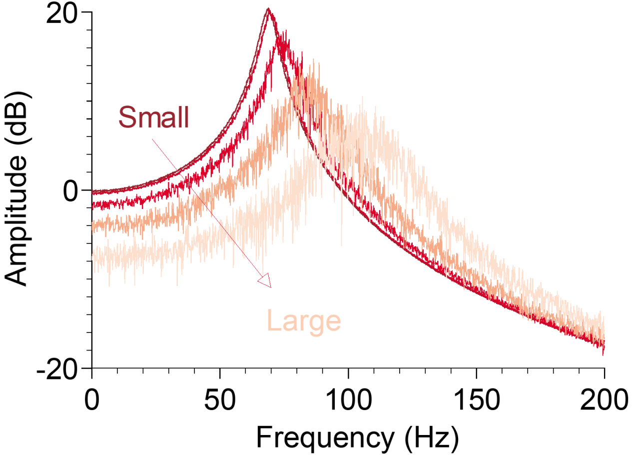
Choose

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{d}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t)$$

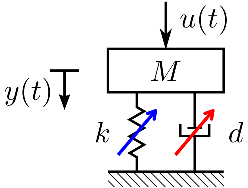
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

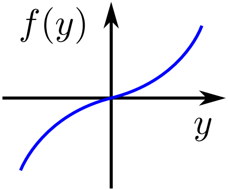
Example: mass-spring-damper



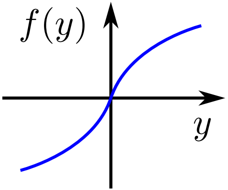
Example: mass-spring-damper

$$M\ddot{y}(t) + f(y(t), \dot{y}(t)) = u(t)$$

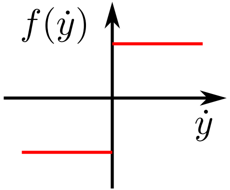




Hardening spring



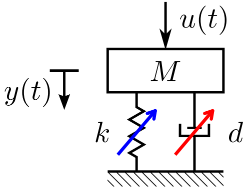
Softening spring



Coulomb friction

Example: mass-spring-damper

$$M\ddot{y}(t)+d\dot{y}(t)+ky(t)+f_{\mathrm{NL}}(y(t),\dot{y}(t)) = u(t)$$



Choose

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \dot{y}(t) \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{d}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -\frac{1}{M} \end{bmatrix} f_{\mathrm{NL}}(y(t),\dot{y}(t))$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Nonlinear discrete-time state-space model

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + f(x(t), u(t)) \\ y(t) &= Cx(t) + Du(t) + g(x(t), u(t)) \end{aligned}$$

Includes block-structured models

- Wiener
- Hammerstein
- Wiener-Hammerstein
- Nonlinear feedback

Nonlinear functions f and g?

- Use basis function expansion
- Many possibilities
- Here: polynomials

Polynomial nonlinear state-space model (PNLSS)

$$x(t+1) = \boxed{A}\, x(t) + \boxed{B}\, u(t) + \boxed{E}\, \zeta(x(t), u(t))$$

$$y(t) = \boxed{C}\, x(t) + \boxed{D}\, u(t) + \boxed{F}\, \eta(x(t), u(t))$$

$\underbrace{A x(t) + B u(t)}_{\text{linear state-space model}} \quad \underbrace{E \zeta(x(t),u(t))}_{\text{polynomials in } x \text{ and } u}$

with e.g. $\zeta(x, u) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_1 u \\ u^3 \\ x_2^2 u \\ \vdots \end{bmatrix}$

Identification of a PNLSS model

$$x(t+1) = Ax(t) + Bu(t) + E\zeta(x(t), u(t))$$
$$y(t) = Cx(t) + Du(t) + F\eta(x(t), u(t))$$

$$\theta = \begin{bmatrix} \text{vec}(A) \\ \text{vec}(B) \\ \text{vec}(C) \\ \text{vec}(D) \\ \text{vec}(E) \\ \text{vec}(F) \end{bmatrix}$$

$$\epsilon(k,\theta) = Y(k,\theta) - Y_{\text{meas}}(k)$$
$$V_{\text{WLS}}(\theta) = \sum_{k=1}^{N_F} \epsilon^H(k,\theta) W(k) \epsilon(k,\theta)$$
$$\hat{\theta} = \arg\min_{\theta} V_{\text{WLS}}$$

Nonlinear in the parameters

- Nonlinear optimization
- Starting values?

Starting values: best linear approximation (BLA)

Random-phase multisine excitations:

$$u_1^{[m]}(t) = \sum_{k=1}^{N_F} A_k \sin(2\pi k f_0 t + \phi_k^{[m]})$$

FRM:

$$G_{BLA}(k) = \frac{1}{M} \sum_{m=1}^{M} Y^{[m]}(k) \left(U^{[m]}(k)\right)^{-1}$$

with e.g. $U^{[m]}(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} U_1^{[m]}(k)$ for two inputs

Nonparametric noise model:

$$C_{\mathrm{BLA}}(k) = C_{\mathrm{NL}}(k) + C_{\mathrm{noise}}(k)$$

Parametric linear model: frequency domain subspace

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

$$\downarrow \text{DFT}$$

$$
\begin{aligned}
e^{j\omega}X(\omega) &= AX(\omega) + BU(\omega) \\
Y(\omega) &= CX(\omega) + DU(\omega)
\end{aligned}
$$

$$\downarrow Y(\omega) = G(\omega)U(\omega)$$

$$
\begin{aligned}
e^{j\omega}X^c(\omega) &= AX^c(\omega) + B \\
G(\omega) &= CX^c(\omega) + D
\end{aligned}
$$

Parametric linear model: frequency domain subspace

$$e^{j\omega}X^c(\omega) = AX^c(\omega) + B$$
$$G(\omega) = CX^c(\omega) + D$$

$$\Downarrow$$

$$\begin{bmatrix} G(\omega) \\ e^{j\omega}G(\omega) \\ e^{j2\omega}G(\omega) \\ \vdots \\ e^{j(r-1)\omega}G(\omega) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{r-1} \end{bmatrix} X^c(\omega) + \begin{bmatrix} D & 0 & \cdots & 0 & 0 \\ CB & D & \ddots & \vdots & \vdots \\ CAB & CB & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & D & 0 \\ CA^{r-2}B & CA^{r-3}B & \cdots & CB & D \end{bmatrix} \begin{bmatrix} I_{n_u} \\ e^{j\omega}I_{n_u} \\ e^{j2\omega}I_{n_u} \\ \vdots \\ e^{j(r-1)\omega}I_{n_u} \end{bmatrix}$$

$\Downarrow$ linear algebra

$(\hat{A}, \hat{B}, \hat{C}, \hat{D})$
E and F initially zero

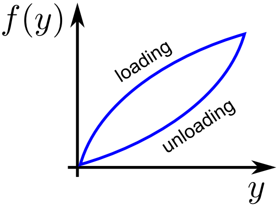
Levenberg-Marquardt optimization

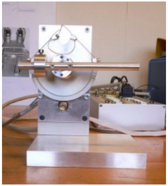
$$\epsilon(k,\theta) = Y(k,\theta) - Y_{\mathrm{meas}}(k)$$

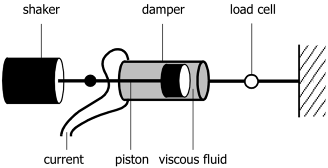
$$\begin{aligned} V_{\mathrm{WLS}}(\theta) &= \sum_{k=1}^{N_F} \epsilon^H(k,\theta) W(k) \epsilon(k,\theta) \\ &= \sum_{k=1}^{N_F} \epsilon_{\mathrm{W}}^H(k,\theta) \epsilon_{\mathrm{W}}(k,\theta) \end{aligned}$$

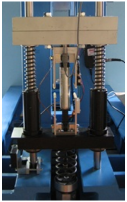
$$\begin{aligned} \theta_{i+1} &= \theta_i + \Delta\theta \\ (J_{\mathrm{W}}^T J_{\mathrm{W}} + \lambda I)\Delta\theta &= -J_{\mathrm{W}}^T \epsilon_{\mathrm{W}} \\ J_{\mathrm{W}} &= \frac{\partial \epsilon_{\mathrm{W}}}{\partial \theta} \end{aligned}$$

Applications

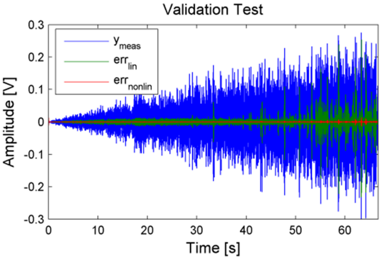


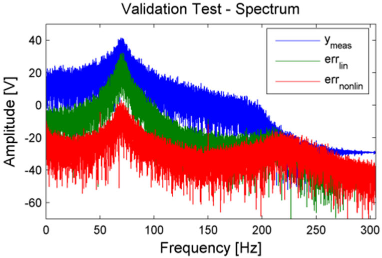






Results Silverbox





Model	Order / degree	RMSE (mV)	# param.
Linear	$n = 2$	14.4	5
Nonlinear	$n = 2$ only $\zeta(x)$, degree = 2, 3	0.57	19
Nonlinear (reduced)	$n = 2$ only $\zeta(x)$, degree = 2, 3	0.46	8

Pros and cons of a PNLSS model

- ✓ Flexibility
- ✓ MIMO
- ✓ Initial estimates

- X Stability
- X Number of parameters
- X Interpretability
- X Extrapolation