Outline

- **Introduction**
  - Uncertainty of The Models
  - MM Description of Uncertainty
  - MM Approach in Three Generic Problems
  - Overview of The Literature
- **Results**
  - A Common Problem
  - MM Approximation
  - MM Estimation
  - MM Control
- **Closure**
  - Summary of Results
  - Perspectives
  - Plan for Doctoral Work

MM = Multiple Model

MM Description of Uncertainty

Single model is not sufficient $\Rightarrow$ select some "typical" models M_i and try to approximate every other situation by *linearly combining* them.

Comments :

- **Strong heuristic appeal**

  M_i are viewed as *operating points* of S, i.e. most of the time S is close to one of M_i

- **Weak theoretical justification**

  undeveloped analysis and synthesis tools, uncovered connections with other models

- **Unexplored practical applicability**

  possible advantages for adaptive control, simpler algorithms and hardware

- **Similar but not equivalent to LPV systems**
  LPV = linear parameter varying

- **Similar but not equivalent to HS**
  HS = hybrid system

MM Approach in 3 Generic Problems

Common problems in system science : approximation, estimation, control.

MM approach :

- **MM approximation** – $\boldsymbol{M(p) \triangleq \sum p_i M_i \approx S}$

  $M(p)$ – MM system, S – ref. model

  Advantage : $\{M_i\}$ simple, e.g. LTI, S complex (uncertain or nonlin.)

- **MM estimation** – $\boldsymbol{\hat{x}(p) \triangleq \sum p_i \hat{x}_i \approx x}$

  $\hat{x}_i$ – given estimates, x – estimated signal

  Advantage : $\hat{x}_i$ designed for LTI systems, $\hat{x}(p)$ apply to more general cases

  e.g. MM Kalman filter

- **MM control** – $\boldsymbol{u(p) = \sum p_i u_i}$

  u_i – given control signals (LTI models),

  $u(p)$ – MM control (uncertain or nonlin. sys.)

  e.g. MM LQG

Overview of The Literature

- **Books :**

  MM Approach to Modeling and Control
  R.M.Smith, T.A.Johanson, T.A.Johansen
  Francis & Taylor, 1997

  Adaptive Estimation and Control: Partitioning Approach
  K. Watanabe, Prentice Hall, 1992

- **Special issue of IJC :** V72, N7/8, May 1999

- **MM first used in :** [Mag65, Lai76, ACD77]

- **MM adaptive control :**
  [NX98, NB97, NBC95, KCC99, LW99, WA99]

- **Biomedical applications :** [YRKB92, HK86]

- **MM estimation and fault detection :**
  [LB96, ZL98, MM95, NL94]

- **Relevant links** in LPV and HS literature

A Common Problem

Given : $A_i \in \mathcal{R}^{m\times n},\ i = 0, 1, \ldots, n_p$

Find : $\boxed{p_{min} = \arg\min_p \|\tilde{A}(\tilde{p}\otimes I_n)\|_F}$

$$\tilde{A} \triangleq [A_0\ A],\ A \triangleq [A_1 \cdots A_{n_p}],\ \tilde{p} \triangleq \begin{bmatrix} -1 \\ p \end{bmatrix},\ p \in \mathcal{R}^{n_p}$$

Solution : $\boxed{p_{min} \in \{F^{\dagger} g + \mathcal{N}(F)\}}$

$$F = \left[\text{tr}\left(A_i^T A_j\right)\right],\ g = \left[\text{tr}\left(A_i^T A_0\right)\right]$$

Proof :

$$\begin{aligned} \|\tilde{A}(\tilde{p}\otimes I_n)\|_F^2 &= \text{tr}\left((\tilde{p}^T\otimes I)\tilde{A}^T\tilde{A}(\tilde{p}\otimes I)\right) \\ &= \tilde{p}^T\left[\text{tr}\left(A_i^T A_j\right)\right]\tilde{p} \\ &= [-1\ p^T]\begin{bmatrix} f & g^T \\ g & F\end{bmatrix}\begin{bmatrix} -1 \\ p\end{bmatrix} \\ &= p^T F p - 2g^T p + f \end{aligned}$$

$$A^T A = \left[A_i^T A_j\right] \geq 0 \xrightarrow{\text{lemma}} F = \left[\text{tr}\left(A_i^T A_j\right)\right] \geq 0$$

$\Rightarrow\ \exists$ minimum, if A full rank, $p_{min} = F^{-1}g$
otherwise, $p_{min} \in \{p \mid F^{\dagger}g + \mathcal{N}(F)\}$. $\square$

We will refer to the common problem as (CP).

Lemma Used in The Proof

Lemma Let $n, n_p \in \mathcal{Z}^+,\ A = A^T \in \mathcal{R}^{n.n_p\times n.n_p}$,

$$A \triangleq \begin{bmatrix} A_{11} & \cdots & A_{1n_p} \\ \vdots & & \vdots \\ A_{n_p1} & \cdots & A_{n_pn_p}\end{bmatrix},\ A_{ij} \in \mathcal{R}^{n\times n}$$

then $A > (\geq) 0 \Rightarrow \left[\text{tr}\left(A_{ij}\right)\right] > (\geq) 0$.

Proof : Let $p \in \mathcal{R}^{n_p}$ and define

$$Z(p) \triangleq p\otimes I = \begin{bmatrix} p_1 I \\ \vdots \\ p_{n_p} I\end{bmatrix} \in \mathcal{R}^{n.n_p\times n}.$$

Note that rank $(Z(p)) = n$ for $\forall p \in \mathcal{R}^{n_p}$ and

$$Z(p)^T A Z(p) = \sum_{i,j} p_i p_j A_{i,j}. \qquad (1)$$

Then

$$\text{rank}\,(Z(p)) = n,\ A > 0\ \Rightarrow Z(p)^T A Z(p) > 0 \Rightarrow \text{tr}\left(Z(p)^T A Z(p)\right) > 0$$

and from (1)

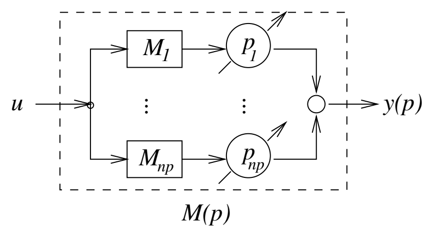
$$\text{tr}\left(\sum_{i,j} p_i p_j A_{i,j}\right) > 0 \Rightarrow \sum_{i,j} p_i p_j \text{tr}\left(A_{i,j}\right) > 0 \Rightarrow p^T\left[\text{tr}\left(A_{ij}\right)\right] p > 0.$$

While p is arbitrary $\left[\text{tr}\left(A_{ij}\right)\right] > 0$. $\square$

MM Approximation

- **Definition**
- **Introductory Example**
- **Solution, Finite Horizon Case**
- **Solution, Infinite Horizon Case**
- **Example**

Definition – Problem Setup



Given ref. model $M_0 : H_0(z)$ and n_p models

$$M_i\ :\ H_i(z) = (A_i, B_i, C_i, D_i),\ i = 1, \ldots, n_p,$$

the **MM approximation** $M(p)$ (of M_0) is

$$M(p) \triangleq \sum_{i=1}^{n_p} p_i M_i\ :\ H(z,p) = \sum_{i=1}^{n_p} p_i H_i(z).$$

The **error of approximation** is

$$\tilde{M}(p) \triangleq M(p) - M_0\ :\ \tilde{H}(z,p) \triangleq H(z,p) - H_0(z).$$

Denote

$$\tilde{h}(t,p) \triangleq \mathcal{Z}^{-1}\left\{\tilde{H}(z,p)\right\}$$

and define the **output error**

$$\tilde{y}(t,p) \triangleq y(t,p) - y(t),$$

$y(t,p)$ – output of $M(p)$, $y(t)$ – output of M_0.

Definition – MM Approximation

Consider the following problems :

- Finite horizon approximation (t_f – horizon length)
  - **AP1**

$$p_F(t_f) = \arg\inf_p \sum_{t=0}^{t_f} \operatorname{tr}\left(\tilde{h}^T(t,p)\tilde{h}(t,p)\right)$$

  - **AP2**

$$p_2(t_f) = \arg\inf_p \sup_{u(t),t=0,\ldots,t_f} \sum_{t=0}^{t_f} \tilde{y}^T(t,p)\tilde{y}(t,p)$$

$$\text{s.t.} \sum_{t=0}^{t_f} u^T(t)u(t) = 1$$

- Infinite horizon approximation
  - **AP3** $\quad p_{\mathcal{H}_2} = \arg\inf_p \|\tilde{H}(z,p)\|_2$
  - **AP4** $\quad p_{\mathcal{H}_\infty} = \arg\inf_p \|\tilde{H}(z,p)\|_\infty$

Solution, Finite Horizon Case

Consider : $u(t), t = 0,1,\ldots,t_f-1$ **and** $x_i(0) = 0$

Output of M_i **:** $Y_i = T_iU$

$$Y_i \triangleq \begin{bmatrix} y_i(0) \\ \vdots \\ y_i(t_f-1) \end{bmatrix}, \; U \triangleq \begin{bmatrix} u(0) \\ \vdots \\ u(t_f-1) \end{bmatrix}, \quad h_i(t) \triangleq \mathcal{Z}^{-1}\{H_i(z)\}$$

$$T_i \triangleq \begin{bmatrix} h_i(0) & & & \\ h_i(1) & h_i(0) & & \\ \vdots & \vdots & \ddots & \\ h_i(t_f-1) & h_i(t_f-2) & \cdots & h_i(0) \end{bmatrix}$$

Output of $M(p)$ **:** $Y(p) = T(p)U$

$$Y(p) = \sum_{i=1}^{n_p} p_iY_i = \underbrace{[T_1 \cdots T_{n_p}]}_{T} \begin{bmatrix} p_1I \\ \vdots \\ p_{n_p}I \end{bmatrix} U = \underbrace{T(p\otimes I)}_{T(p)}U = T(p)U$$

Output error : $\tilde{Y}(p) = \tilde{T}(p)U$

$$\tilde{Y}(p) = Y(p) - Y = (T(p) - T_0)U$$

$$= \underbrace{[T_0 \; T_1 \cdots T_{n_p}]}_{\tilde{T}} \begin{bmatrix} -I \\ p\otimes I \end{bmatrix} U = \underbrace{\tilde{T}(\tilde{p}\otimes I)}_{\tilde{T}(p)}U = \tilde{T}(p)U$$

Solution, Finite Horizon Case – AP2

AP2 : $\inf_p \sup_{U,\|U\|_2=1} \|\tilde{Y}(p)\|_2 = \inf_p \|\tilde{T}(p)\|_2$

$$\boxed{p_2(t_f) = \arg\inf_p \|\tilde{T}(p)\|_2}$$

The solution is SDP problem
(n_p+1 dec. var., $t_f(n_u+n_y)$ size LMI)

$$p_2(t_f) = \arg\inf t \text{ s.t. } \begin{bmatrix} tI & \tilde{T}(\tilde{p}\otimes I) \\ \left(\tilde{T}(\tilde{p}\otimes I)\right)^T & tI \end{bmatrix} \geq 0$$

Not justified but easier to solve is

$$\boxed{p_{FT}(t_f) = \arg\inf_p \|\tilde{T}(p)\|_F}$$

This is a CP with $A_i = T_i \;\Rightarrow p_{FT}(t_f) = F^{-1}g$.

$$F = \left[\operatorname{tr}\left(T_i^TT_j\right)\right], \quad g = \left[\operatorname{tr}\left(T_i^TT_0\right)\right].$$

Substitute T_i for H_i, the $t_f \times 1$ block Toeplitz matrix

$$F = \left[\operatorname{tr}\left(H_i^TH_j\right)\right], \quad g = \left[\operatorname{tr}\left(H_i^TH_0\right)\right].$$

You get $p_{FH}(t_f) = p_F(t_f)$. Explained next.

Solution, Finite Horizon Case – AP1

AP1 : $\boxed{\inf_p \sum_{t=0}^{t_f} \operatorname{tr}\left(\tilde{h}^T(t,p)\tilde{h}(t,p)\right)}$

Define

$$\tilde{H}(p) \triangleq \begin{bmatrix} \tilde{h}(0,p) \\ \vdots \\ \tilde{h}(t_f-1,p) \end{bmatrix} \Rightarrow \sum_{t=0}^{t_f} \tilde{h}^T(t,p)\tilde{h}(t,p) = \tilde{H}(p)^T\tilde{H}(p)$$

Note that

$$\tilde{H}(p) = H(p) - H_0 = H(p\otimes I) - H_0 = \tilde{H}(\tilde{p}\otimes I)$$

$$H = [H_1 \cdots H_{n_p}], \; \tilde{H} = [H_0 \; H]$$

Then

$$p_F(t_f) = \arg\inf_p \operatorname{tr}\left(\tilde{H}^T(p)\tilde{H}(p)\right) = \arg\inf_p \|\tilde{H}(\tilde{p}\otimes I)\|_F$$

AP1 is a CP with $A_i = H_i \;\Rightarrow$

$$\boxed{p_F(t_f) = F^{-1}g}$$

$$F = \left[\operatorname{tr}\left(H_i^TH_j\right)\right], \quad g = \left[\operatorname{tr}\left(H_i^TH_0\right)\right]$$

Solution, Infinite Horizon Case – AP3

AP3 : $\boxed{p_{\mathcal{H}_2} = \arg\inf_p \|\tilde{H}(z,p)\|_2}$

First, note that $H(z,p) = \tilde{H}(z)(\tilde{p} \otimes I)$

$$H(z,p) = \sum_{i=1}^{n_p} p_i H_i(z) = [H_1(z) \cdots H_{n_p}(z)](p \otimes I) = H(z)(p \otimes I)$$

$$\tilde{H}(z,p) = H(z,p) - H_0(z) = \underbrace{[H_0(z) H(z)]}_{\tilde{H}(z)} \begin{bmatrix} -I \\ p \otimes I \end{bmatrix} = \tilde{H}(z)(\tilde{p} \otimes I)$$

Next, determine the **impulse response of $\tilde{H}(z,p)$**

$$\tilde{h}(t,p) = \mathcal{Z}^{-1}\left\{\tilde{H}(z,p)\right\} = \tilde{h}(t)(\tilde{p} \otimes I)$$

Then

$$p_{\mathcal{H}_2} = \arg\inf_p \sum_{t=0}^{\infty} \text{tr}\left(\tilde{p}^T \otimes I\right) \tilde{h}^T(t)\tilde{h}(t)(\tilde{p} \otimes I) = \arg\inf_p \|\sqrt{\tilde{S}}(\tilde{p} \otimes I)\|_F$$

$\tilde{S} \triangleq \left[S_{ij}\right],\ S_{ij} \triangleq \sum_{t=0}^{\infty} \tilde{h}_i^T(t)\tilde{h}_j(t)$

AP3 is a CP with $A_i = \sqrt{\tilde{S}_{ii}}$ $\Rightarrow$ $\boxed{p_{\mathcal{H}_2} = F^{-1}g}$

$$F = \left[\text{tr}\left(S_{ij}\right)\right], \quad g = [\text{tr}\left(S_{i0}\right)]$$

How to compute S_{ij}?

Solution, Infinite Horizon Case – Computation of S_{ij}

$$\boxed{S_{ij} = \sum_{t=0}^{\infty} \tilde{h}_i^T(t)\tilde{h}_j(t)}$$

$$\tilde{h}(t) \triangleq [h_0(t)\ h_1(t) \cdots h_{n_p}(t)],\ \tilde{S} \triangleq [S_{ij}] \Rightarrow \tilde{S} = \sum_{t=0}^{\infty} \tilde{h}^T(t)\tilde{h}$$

Define the system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, where

$\tilde{A} \triangleq \text{diag}\{A_0, A_1, \ldots, A_{n_p}\},\ \ \tilde{B} \triangleq \text{diag}\{B_0, \ldots, B_{n_p}\}$

$\tilde{C} \triangleq [C_0\ C_1 \cdots C_{n_p}],\ \ \tilde{D} \triangleq [D_0\ D_1 \cdots D_{n_p}]$

$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a **realization of $\tilde{h}(t)$** $\Rightarrow$

$$\begin{aligned} \tilde{S} &= \sum_{t=0}^{\infty} \tilde{h}^T(t)\tilde{h}(t) \\ &= \sum_{t=0}^{\infty} \left(\tilde{C}\tilde{A}^t\tilde{B}\right)^T \left(\tilde{C}\tilde{A}^t\tilde{B}\right) + \tilde{D}^T\tilde{D} \\ &= \tilde{B}^T \sum_{t=0}^{\infty} \left(\tilde{A}^{Tt}\tilde{C}^T\tilde{C}\tilde{A}^t\right)\tilde{B} + \tilde{D}^T\tilde{D} \end{aligned}$$

$$\boxed{\tilde{S} = \tilde{B}^T\tilde{W}\tilde{B} + \tilde{D}^T\tilde{D}}$$

$\tilde{W}$ is the **solution of the DT Lyapunov eqn.**

$$\tilde{W} - \tilde{A}^T\tilde{W}\tilde{A} = \tilde{C}^T\tilde{C}$$

Solution, Infinite Horizon Case – AP4

AP4 : $\boxed{p_{\mathcal{H}_\infty} = \arg\inf_p \|\tilde{H}(z,p)\|_\infty}$

Bounded-real lemma for DT sys.

$$\|\tilde{H}(z,p)\|_\infty < \gamma \Leftrightarrow \exists X = X^T > 0 \text{ s.t.}$$

$$\begin{bmatrix} \tilde{A}^T X\tilde{A} - X & \tilde{A}^T X\tilde{B} & \tilde{C}^T(p) \\ \tilde{B}^T X\tilde{A} & \tilde{B}^T X\tilde{B} - \gamma I & \tilde{D}^T(p) \\ \tilde{C}(p) & \tilde{D}(p) & -\gamma I \end{bmatrix} < 0 \qquad (3)$$

where $(\tilde{A}, \tilde{B}, \tilde{C}(p), \tilde{D}(p))$ is a realization of $\tilde{H}(z,p)$

$$\tilde{A} \triangleq \begin{bmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_{n_p} \end{bmatrix} \qquad \tilde{B} \triangleq \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{n_p} \end{bmatrix}$$

$$\tilde{C}(p) \triangleq [-C_0\ p_1C_1 \cdots p_{n_p}C_{n_p}] \qquad \tilde{D}(p) \triangleq \sum_{i=1}^{n_p} p_i D_i$$

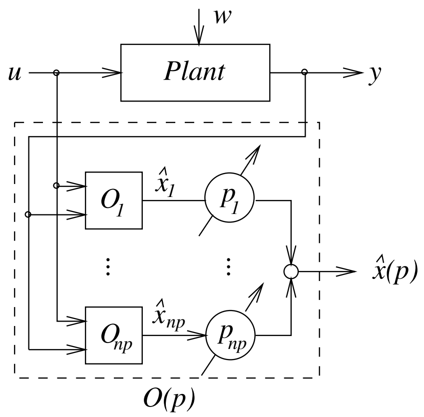
(3) is an LMI in the design var. $X,\ p,\ \gamma\ \Rightarrow$
AP4 is an SDP problem

$\tilde{n}(\tilde{n}+1)/2 + n_p + 1$ – dec. var., $(\tilde{n} \triangleq (n_p+1).n)$
$\tilde{n} + n_u + n_y$ – size LMI

MM Estimation

- **Definition**
- **Description of the Closed-loop System**
- **Time-invariant Analysis**
- **Self-tuning Algorithm**
- **Example**

Definition – Problem Setup



M_0 – **nominal plant,** $\{M_i\}_{i=1}^{n_p}$ – **set of models**

$$M_i \ : \ H_i(z) = (A_i, B_i, C_i, 0) \ \ i = 0, 1, \ldots, n_p$$

Set of observers : $O_i \leftrightarrow M_i$ (estimate $\hat{x}_i(t)$)

MM observer : $O(p) = \sum_{i=1}^{n_p} p_i O_i$

inputs : $\begin{bmatrix} u \\ y \end{bmatrix}$, output : $\hat{x}(t,p) \triangleq \sum_{i=1}^{n_p} p_i \hat{x}_i(t)$

Definition – MM Estimation

Error of estimation : $\tilde{x}(t,p) \triangleq \hat{x}(t,p) - x(t)$

Output error of estimation

$$\tilde{y}(t,p) \triangleq \hat{y}(t,p) - y(t)$$

$\hat{y}(t,p) = \sum_{i=1}^{n_p} p_i C_i \hat{x}_i$ – predicted output

Define : $V_{\tilde{x}}(p) \triangleq \mathbf{E}\left\{\tilde{x}\tilde{x}^T\right\}, \ V_{\tilde{y}}(p) \triangleq \mathbf{E}\left\{\tilde{y}\tilde{y}^T\right\}$

$$\hat{V}_{\tilde{y}}(t,p) \triangleq \frac{1}{t-1} \sum_{i=0}^{t} \tilde{y}(i)\tilde{y}^T(i)$$

MM estimation problems

- Time-invariant analysis
  - **EP1** $\quad p_{\tilde{x}} = \arg\inf_p \mathrm{tr}\left(V_{\tilde{x}}(p)\right)$
  - **EP2** $\quad p_{\tilde{y}} = \arg\inf_p \mathrm{tr}\left(V_{\tilde{y}}(p)\right)$
- Self-tuning state estimator
  - **EP3** $\quad p_{st}(t) = \arg\inf_p \mathrm{tr}\left(\hat{V}_{\tilde{y}}(t,p)\right)$

    $\hat{V}_{\tilde{y}}(t,p)$ – real-time estimate of $V_{\tilde{y}}$

Time-invariant Analysis – EP1

EP1 : $\boxed{\inf_p \mathrm{tr}\left(V_{\tilde{x}}(p)\right)}$

Covariance of the augmented state $\tilde{x}$

$$V_{\tilde{x}} - \tilde{A}^T V_{\tilde{x}} \tilde{A} = \tilde{B} \underbrace{\begin{bmatrix} V_u & & \\ & V_{w_d} & \\ & & V_{w_n} \end{bmatrix}}_{V_{\tilde{u}}} \tilde{B}^T, \ \ V_{\tilde{x}} = \begin{bmatrix} V_x & V_{x\hat{x}} \\ V_{\hat{x}x} & V_{\hat{x}} \end{bmatrix}$$

Error of estimation

$$\tilde{x}(t,p) \triangleq \hat{x}(t,p) - x(t) = C_{\tilde{x}}(p)\tilde{x}(t)$$

$$C_{\tilde{x}}(p) = [-I \ \ p_1 I \cdots p_{n_p} I] = \underbrace{[-1 \ \ p^T]}_{\tilde{p}^T} \otimes I = \tilde{p}^T \otimes I$$

Covariance of the error of estimation

$$V_{\tilde{x}}(p) = \mathbf{E}\left\{\tilde{x}\tilde{x}^T\right\} = (\tilde{p}^T \otimes I) V_{\tilde{x}} (\tilde{p} \otimes I)$$

$$\mathrm{tr}\left(V_{\tilde{x}}(p)\right) = \|\sqrt{V_{\tilde{x}}}(\tilde{p} \otimes I)\|_F$$

EP1 is a CP with
$A_0 = \sqrt{V_x}, \ A_i = \sqrt{V_{\hat{x}_i}}, \ i = 1, \ldots, n_p$

$$\boxed{p_{\tilde{x}} = F^{-1} g}$$

$$F \triangleq \left[\mathrm{tr}\left(V_{\hat{x}_i \hat{x}_j}\right)\right], \ \ g \triangleq \left[\mathrm{tr}\left(V_{\hat{x}_i x}\right)\right]$$

Time-invariant Analysis – EP2

EP2 : $\boxed{\inf_p \mathrm{tr}\left(V_{\tilde{y}}(p)\right)}$

Output error of the i**-th KF** $\tilde{y}_i(t)$

$$\tilde{y}_i(t) \triangleq \underbrace{C_i \hat{x}_i(t)}_{\hat{y}_i(t)} - y(t) = \hat{y}_i(t) - y(t)$$

Output error of the MMKF $\tilde{y}(t,p)$

$$\tilde{y}(t,p) \triangleq \hat{y}(t,p) - y(t) = \hat{Y}(t)p - y(t) = \underbrace{[y(t) \ \ \hat{Y}(t)]}_{\tilde{Y}(t)} \begin{bmatrix} -1 \\ p \end{bmatrix} = \tilde{Y}(t)\tilde{p}$$

$$\hat{y}(t,p) \triangleq \sum_{i=1}^{n_p} p_i \hat{y}_i(t) = \underbrace{[\hat{y}_1(t) \cdots \hat{y}_{n_p}(t)]}_{\hat{Y}(t)} p = \hat{Y}(t)p$$

Covariance of the output error

$$\mathrm{tr}\left(V_{\tilde{y}}(p)\right) = \mathbf{E}\left\{\tilde{y}^T(t,p)\tilde{y}(t,p)\right\} = \mathbf{E}\left\{\tilde{p}^T \tilde{Y}^T(t)\tilde{Y}(t)\tilde{p}\right\}$$

$$\inf_p \mathrm{tr}\left(V_{\tilde{y}}(p)\right) = (\tilde{p}^T \otimes 1) V_{\tilde{Y}} (\tilde{p} \otimes 1) = \|\sqrt{V_{\tilde{Y}}}(\tilde{p} \otimes 1)\|_F$$

EP2 is a CP with
$A_0 = \sqrt{V_y}, \ A_i = \sqrt{V_{\hat{y}_i}}, \ i = 1, \ldots, n_p$

$$\boxed{p_{\tilde{y}} = F^{-1} g}$$

$$F \triangleq \left[\mathrm{tr}\left(V_{\hat{y}_i \hat{y}_j}\right)\right], \ \ g \triangleq \left[\mathrm{tr}\left(V_{\hat{y}_i y}\right)\right]$$

Self-tuning Algorithm – EP3

EP3 : $\boxed{\inf_p \mathrm{tr}\left(\widehat{V}_{\tilde{y}}(t,p)\right)}$

Because of stationarity

$$V_{\tilde{y}}(p) = \lim_{t_f \to \infty} \frac{1}{t_f} \sum_{t=0}^{t_f} \tilde{y}(p,t)\tilde{y}^T(p,t)$$

This awolls real-time estimation of $V_{\tilde{y}}$

$$\widehat{V}_{\tilde{y}}(t,p) = \frac{1}{t} \sum_{i=0}^{t} \tilde{y}(i,p)\tilde{y}^T(i,p)$$

$$\widehat{V}_{\tilde{y}}(t,p) \to V_{\tilde{y}}(p) \text{ as } t \to \infty$$

At time t we use $\widehat{V}_{\tilde{y}}(t,p)$ as a substitute of $V_{\tilde{y}}(p)$.

EP3 is a CP with
$A_0 = \sqrt{\hat{V}_y(t)},\ A_i = \sqrt{\hat{V}_{\hat{y}_i}(t)},\ i = 1,\ldots,n_p$

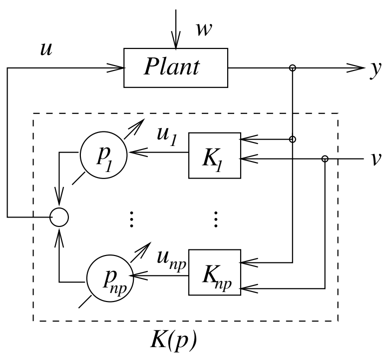
$$\boxed{p_{st}(t) = F^{-1}(t)g(t)}$$

$$F(t) \triangleq \left[\mathrm{tr}\left(\widehat{V}_{\hat{y}_i\hat{y}_j}(t)\right)\right],\ g(t) \triangleq \left[\mathrm{tr}\left(\widehat{V}_{\hat{y}_i y}(t)\right)\right]$$

MM Control

- Definition
- Description of the Closed-loop System
- Time-invariant Analysis
- Example

Definition



Under the assumptions made before.

Cost function $Q = Q^T \geq 0,\ R = R^T > 0$

$$J(p) = \mathbf{E}\left\{x^T(t,p)Qx(t,p) + u^T(t,p)Ru(t,p)\right\}$$

i-th LQG controller $K_i(z)$

$K_{r,i}$ – LQR gain for $(A_i, B_{u,i})$
$K_{f,i}$ – KF gain for $(A_i, B_{w_d,i}, C_i)$

MM controller : $\sum_{i=1}^{n_p} p_i K_i(z)$

MM contol prob. (CP1) : $\boxed{p_{min} = \arg\inf_p J(p)}$

Time-invariant Analysis

Expected value and covariance of $\tilde{x}$

$$\mathbf{E}\{\tilde{x}\} = (I - \tilde{A})^{-1}\tilde{B}\mathbf{E}\{\tilde{u}\}$$

$$V_{\tilde{x}} = \tilde{A}^T V_{\tilde{x}} \tilde{A} + \tilde{B} \underbrace{\begin{bmatrix} V_u & & \\ & V_{w_d} & \\ & & v_{w_n} \end{bmatrix}}_{V_{\tilde{u}}} \tilde{B}^T,\ V_{\tilde{x}} = \begin{bmatrix} V_x & V_{x\hat{x}} \\ V_{\hat{x}x} & V_{\hat{x}} \end{bmatrix}$$

Performance index (v=const.)

$$\begin{aligned} J(p) = & \quad \mathbf{E}\left\{x^T Qx + u^T Ru\right\} \\ = & \ p^T Fp + 2g^T p + v^T Rv + \mathrm{tr}\left(QV_x\right) \end{aligned} \quad (4)$$

where

$$F = \left[tr(K_{r,i}{}^T RK_{r,j}V_{\hat{x}_i\hat{x}_j})\right],\quad g^T = v^T R\mathbf{E}\{U\}$$

$$U = [K_{r,1}\hat{x}_1 \cdots K_{r,n_p}\hat{x}_{n_p}]$$

Minimization of $J(p)$

$\mathrm{tr}\left(QV_x\right)$ in (4) makes CP1 hard.

Currently we do not have solution.