

# Innovation-based subspace identification in open- and closed-loop

Guillaume Mercère, Ivan Markovsky and José A. Ramos

**Abstract**—The applicability of subspace-based system identification methods highly depends on the disturbances acting on the system. It is well-known, *e.g.*, that the standard implementations of the MOESP, N4SID or CVA algorithms yield biased estimates when closed-loop noisy data is considered. In this paper, we suggest pre-estimating the innovation term from the available data in order to bypass this difficulty. By doing so, the subspace-based identification problem can be written as a deterministic problem for which efficient methods exist. When the system description does not belong to the model class, a structured least-squares solution is proposed. The performance of the methods is illustrated through the study of simulation examples.

## I. INTRODUCTION

In almost three decades, the subspace-based state-space system identification (4SID) techniques have proved their effectiveness in estimating accurate models of linear systems [1], [2], [3]. These 30 years have given rise to the development of many algorithms yielding consistent estimates of discrete-time or continuous-time linear time-invariant (LTI), linear parameter-varying, and bi-linear systems working under open-loop or closed-loop conditions and disturbed by input, process and/or output noises [4], [5], [6], [7]. The robustness of the tools used in subspace-based identification is probably the main reason why the 4SID techniques are considered as a good alternative and a good initial step of the standard maximum likelihood methods [8] for state-space model identification.

As described in [9, Section 2.4], the 4SID methods are multi-step techniques consisting of a pre-estimation step requiring high-order ARX model estimates followed by least-squares regressions to extract the system's state sequence or observability subspace from which the state-space parameters can be estimated. Following this observation, the problem of estimating accurate black-box discrete-time LTI models under open-loop or closed-loop conditions in a subspace-based identification framework is revisited hereafter through a unified formulation. Inspired by the developments in [10], [11], we suggest a two-step approach.

First, in order to bypass the standard noise related issues in subspace-based identification (bias, correlation with the

input signals, ...), we estimate the innovation term using a high-order ARX model [9], [12]. Second, we describe the data equations obtained for the standard innovation form [4] as well as the predictor form [13] as a structured least-squares regression problem. We focus on the lower triangular Toeplitz structure of matrices. Using the high-order ARX model [10], [13], [14], [9], [12] is the keystone for getting a consistent (in the Rouché-Capelli theorem sense [15]) structured linear least-squares regression problem. This change of viewpoint, *i.e.*, focusing on the innovation term instead of the Markov parameters directly, is the main difference between the method proposed in this paper and the ones in [13], [14]. This change allows us to select any deterministic subspace-based technique for the actual estimation of state-space model parameters. In this paper, specific attention is given to data-driven algorithm [16] for systems working under open-loop and/or closed-loop conditions. Such an approach further allows us to incorporate prior information about the system. However, incorporation of prior knowledge will be treated elsewhere.

The paper is organized as follows. After the introduction of the main notations in Section II, the system identification problem is defined in Section III. The main contribution of the paper, *i.e.*, the description of the open-loop and closed-loop 4SID algorithms, is given in Section IV. These new methods are then compared on simulation examples with the state space auto-regressive exogenous (SSARX) technique [14] and the standard N4SID methods in Section V. Section VI concludes the paper.

## II. NOTATIONS

For any vector  $\mathbf{r}(t) \in \mathbb{R}^{n_r}$  and natural number  $\ell \in \mathbb{N}_*^+$ , we define

- the finite past stacked vector  $\mathbf{r}_\ell^-(t) \in \mathbb{R}^{\ell n_r \times 1}$

$$\mathbf{r}_\ell^-(t) = [\mathbf{r}^\top(t-\ell) \quad \cdots \quad \mathbf{r}^\top(t-1)]^\top, \quad (1)$$

- the finite future stacked vector  $\mathbf{r}_\ell^+(t) \in \mathbb{R}^{\ell n_r \times 1}$

$$\mathbf{r}_\ell^+(t) = [\mathbf{r}^\top(t) \quad \cdots \quad \mathbf{r}^\top(t+\ell-1)]^\top. \quad (2)$$

By having access to these finite stacked vectors, the past and future Hankel matrices (resp.  $\mathbf{R}_{\ell,M}^-(t) \in \mathbb{R}^{\ell n_r \times M}$  and  $\mathbf{R}_{\ell,M}^+(t) \in \mathbb{R}^{\ell n_r \times M}$ ) are defined as

$$\mathbf{R}_{\ell,M}^-(t) = [\mathbf{r}_\ell^-(t) \quad \cdots \quad \mathbf{r}_\ell^-(t+M-1)], \quad (3)$$

$$\mathbf{R}_{\ell,M}^+(t) = [\mathbf{r}_\ell^+(t) \quad \cdots \quad \mathbf{r}_\ell^+(t+M-1)]. \quad (4)$$

Using the state space parameters  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{D}$ , for  $\ell \geq n_x$ , we defined the extended observability matrix

$$\mathbf{\Omega}_\ell(\mathbf{A}, \mathbf{B}) = [\mathbf{A}^{\ell-1}\mathbf{B} \quad \cdots \quad \mathbf{A}\mathbf{B} \quad \mathbf{B}], \quad (5)$$

This work was not supported by any organization.

Guillaume Mercère (Corresponding author) is with Poitiers University, Laboratoire d'Informatique et d'Automatique pour les Systèmes, 2 rue P. Brousse, bâtiment B25, TSA 41105, 86073 Poitiers cedex 9, France [guillaume.mercere@univ-poitiers.fr](mailto:guillaume.mercere@univ-poitiers.fr)

Ivan Markovsky is with Vrije Universiteit Brussel, Department ELEC, Pleinlaan 2, 1050 Brussels, Belgium [Ivan.Markovsky@vub.ac.be](mailto:Ivan.Markovsky@vub.ac.be)

José A. Ramos is with Nova Southeastern University, Farquhar College of Arts and Sciences, Division of Mathematics, Science, and Technology, 3301 College Avenue, Fort Lauderdale, FL 33314, USA [jr1284@nova.edu](mailto:jr1284@nova.edu)

the extended reversed controllability matrix

$$\Gamma_\ell(\mathbf{A}, \mathbf{B}) = \begin{bmatrix} \mathbf{B} \\ \mathbf{BA} \\ \vdots \\ \mathbf{BA}^{\ell-1} \end{bmatrix}, \quad (6)$$

and the block lower-triangular Toeplitz matrix

$$\mathbf{H}_\ell(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) = \begin{bmatrix} \mathbf{D} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{CB} & \mathbf{D} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{CA}^{\ell-2}\mathbf{B} & \cdots & \mathbf{CB} & \mathbf{D} \end{bmatrix}. \quad (7)$$

### III. PROBLEM FORMULATION

In this paper, we consider finite-dimensional, discrete-time, LTI dynamical systems, described by a state-space representation of the form

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t), \quad (8a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t), \quad (8b)$$

where  $\mathbf{x}(t) \in \mathbb{R}^{n_x}$  is the state vector,  $\mathbf{u}(t) \in \mathbb{R}^{n_u}$  is the input vector,  $\mathbf{y}(t) \in \mathbb{R}^{n_y}$  is the output vector,  $\mathbf{v}(t) \in \mathbb{R}^{n_y}$  is the output measurement noise vector and  $\mathbf{w}(t) \in \mathbb{R}^{n_x}$  is the process noise vector.  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$  are the state-space parameters of the system. We consider systems working under open-loop or closed-loop conditions

The following standard assumptions are made in the sequel.

*Assumption 1:* The noise terms  $\mathbf{v}$  and  $\mathbf{w}$  in the LTI state-space representation (8) are independent zero-mean white Gaussian noises with finite covariance matrices, *i.e.*,

$$\mathbb{E} \left\{ \begin{bmatrix} \mathbf{v}(k) \\ \mathbf{w}(k) \end{bmatrix} \begin{bmatrix} \mathbf{v}^\top(\ell) & \mathbf{w}^\top(\ell) \end{bmatrix} \right\} = \begin{bmatrix} \mathbf{R} & \mathbf{S}^\top \\ \mathbf{S} & \mathbf{Q} \end{bmatrix} \delta_{k\ell},$$

where  $\delta_{k\ell}$  is the Kronecker delta function.

*Assumption 2:* The LTI state-space system (8) is minimal, *i.e.*,  $(\mathbf{A}, \mathbf{C})$  is observable and  $(\mathbf{A}, [\mathbf{B}, \mathbf{Q}^{1/2}])$  is controllable.

*Assumption 3:* The input  $\mathbf{u}$  is assumed to be quasi-stationary and exciting of sufficient order [17].

*Assumption 4:* The feedback loop is assumed to contain at least one sample delay, *i.e.*, the controller or the process has no direct feed-through [18], [19].

In the following, when closed-loop conditions are considered, we will assume that  $\mathbf{D} = \mathbf{0}$ . Theoretically, Assumption 4 ensures the identifiability of the transfer function of the plant  $\mathcal{S}$  (see [18], [19] for a discussion about this property and the following consequences).

*Assumption 5:* The closed-loop system is assumed to be asymptotically stable.

By Assumptions 1 and 2, the state-space model (8) is equivalent to the innovation form

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathcal{K}\mathbf{e}(t), \quad (9a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{e}(t), \quad (9b)$$

where  $\mathcal{K}$  is the steady-state Kalman filter gain and  $\mathbf{e}$  is called the innovation vector [8]. As a consequence of Assumption 1,

the innovation is a stationary, zero-mean white Gaussian noise with covariance  $\mathbf{R}_e$ . Under open-loop condition,  $\mathbf{u}(k)$  and  $\mathbf{e}(k)$  are uncorrelated. Under closed-loop conditions, Assumption 4 guarantees that the innovation sequence  $\mathbf{e}(j)$  and the input  $\mathbf{u}(k)$  are uncorrelated  $\forall j \geq k$  [20].

When open-loop systems (9) are considered, straightforward recursion yields

$$\mathbf{Y}_{f,M}^+ = \Gamma_f(\mathbf{A}, \mathbf{C})\mathbf{x}_M^+(t) + \mathbf{H}_f^{ol,u}\mathbf{U}_{f,M}^+ + \mathbf{N}_{f,M}^+, \quad (10)$$

where

$$\mathbf{H}_f^{ol,u} = \mathbf{H}_f(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}), \quad (11a)$$

$$\mathbf{N}_{f,M}^+ = \underbrace{\mathbf{H}_f(\mathbf{A}, \mathcal{K}, \mathbf{C}, \mathbf{I}_{n_y})}_{\mathbf{H}_f^{ol,e}} \mathbf{E}_{f,M}^+. \quad (11b)$$

Under closed-loop conditions, in order to bypass the problem of correlation between the innovation term and the input signal, most techniques [13], [14], [9], [12], [21], [22] we use a different data equation obtained by considering similar recursions as used in open-loop case but, now, from the predictor state-space model defined as follows

$$\mathbf{x}(t+1) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{B}}\mathbf{u}(t) + \mathcal{K}\mathbf{y}(t), \quad (12a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{e}(t), \quad (12b)$$

where

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathcal{K}\mathbf{C}, \quad \tilde{\mathbf{B}} = \mathbf{B} - \mathcal{K}\mathbf{D}. \quad (13)$$

We have

$$\mathbf{Y}_{f,M}^+ = \Gamma_f(\tilde{\mathbf{A}}, \mathbf{C})\mathbf{x}_M^+(t) + \mathbf{H}_f^{cl,u}\mathbf{U}_{f,M}^+ + \mathbf{H}_f^{cl,y}\mathbf{Y}_{f,M}^+ + \mathbf{E}_{f,M}^+, \quad (14)$$

where

$$\mathbf{H}_f^{cl,u} = \mathbf{H}_f(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \mathbf{C}, \mathbf{D}), \quad (15a)$$

$$\mathbf{H}_f^{cl,y} = \mathbf{H}_f(\tilde{\mathbf{A}}, \mathcal{K}, \mathbf{C}, \mathbf{0}). \quad (15b)$$

The main difficulties encountered in subspace-based identification are the following.

- In Eq. (10) and Eq. (14), the matrices  $\Gamma_f(\mathbf{A}, \mathbf{C})$ ,  $\Gamma_f(\tilde{\mathbf{A}}, \mathbf{C})$ ,  $\mathbf{H}_f^{ol,u}$ ,  $\mathbf{H}_f^{cl,u}$ ,  $\mathbf{H}_f^{cl,y}$  as well as the state sequence  $\mathbf{x}_M^+$  are unknown. Because of products of unknowns, a least-squares solution cannot be directly implemented.
- Both equations contain a noise term, unknown by construction, which is either colored or correlated with the input signal.

We circumvent the first issue by replacing the state sequence in Eq. (10) and Eq. (14) with linear combinations of past input and output signals [23], [24]. For the second problem, instead of introducing a user-defined instrumental variable (as done in [25], [26], [27]) or pre-estimating  $\mathbf{H}_f^{cl,u}$  and  $\mathbf{H}_f^{cl,y}$  (as done in [13], [14], [12]), we suggest pre-estimating the innovation term as considered (as done in [11]). This solution can indeed be used independent of the considered practical conditions (open-loop or closed-loop). Both solutions are detailed in the next section. Notice right now

that the corresponding algorithms *will be called SSinnov1* in Section V.

#### IV. OPEN AND CLOSED LOOP SUBSPACE-BASED IDENTIFICATION METHODS

##### A. From a Stochastic to a Deterministic Identification Problem

The state sequence  $\mathbf{x}_M^+$  in Eq. (10) and Eq. (14) can be described as linear combinations of past data [28], [25], [29], [30]. This is true also closed-loop condition [13], [14], [12]. More precisely, by starting from the predictor form (12), with standard recursions, it is clear that, for any user-defined  $p \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbf{x}(t) &= \tilde{\mathbf{A}}^p \mathbf{x}(t-p) \\ &+ \Omega_p(\tilde{\mathbf{A}}, \mathcal{K}) \mathbf{y}_p^-(t) + \Omega_p(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \mathbf{u}_p^-(t). \end{aligned} \quad (16)$$

From this equation, by assuming that

*Assumption 6:* The LTI state-space form (9) is strict minimum phase *i.e.*,

$$\lambda_{\max}(\mathbf{A} - \mathcal{K}\mathbf{C}) < 1,$$

the state sequence approximation  $\bar{\mathbf{x}}$  defined as follows

$$\bar{\mathbf{x}}(t) = \Omega_p(\tilde{\mathbf{A}}, \mathcal{K}) \mathbf{y}_p^-(t) + \Omega_p(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \mathbf{u}_p^-(t), \quad (17)$$

can be viewed as the the optimal linear estimate of  $\mathbf{x}$  (in the mean-square error sense) given  $\mathbf{u}_p^-(t)$  and  $\mathbf{y}_p^-(t)$  [24], [26]. By using this state approximation, Eq. (10) and Eq. (14) become

$$\begin{aligned} \mathbf{Y}_{f,M}^+ &= \Gamma_f(\mathbf{A}, \mathbf{C}) \Omega_p(\tilde{\mathbf{A}}, [\mathcal{K} \quad \tilde{\mathbf{B}}]) \mathbf{Z}_{p,M}^- \\ &+ \mathbf{H}_f^{ol,u} \mathbf{U}_{f,M}^+ + \mathbf{N}_{f,M}^+, \end{aligned} \quad (18a)$$

$$\begin{aligned} \mathbf{Y}_{f,M}^+ &= \Gamma_f(\tilde{\mathbf{A}}, \mathbf{C}) \Omega_p(\tilde{\mathbf{A}}, [\mathcal{K} \quad \tilde{\mathbf{B}}]) \mathbf{Z}_{p,M}^- \\ &+ \mathbf{H}_f^{cl,u} \mathbf{U}_{f,M}^+ + \mathbf{H}_f^{cl,y} \mathbf{Y}_{f,M}^+ + \mathbf{E}_{f,M}^+, \end{aligned} \quad (18b)$$

respectively, where  $\mathbf{Z}_{p,M}^-$  is defined via Eq. (3) with

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{bmatrix} \in \mathbb{R}^{n_y+n_u}. \quad (19)$$

In [23], [31], [14], [12], this equation is used to estimate the unknown block matrices (*e.g.*,  $\Gamma_f(\mathbf{A}, \mathbf{C})$  and  $\mathbf{H}_f^{ol,u}$  when Eq. (18a) is considered). Herein, we focus on the estimation of the innovation. If the noise is known, the aforementioned issues related to the disturbances acting on the system are circumvented. In order to reach this goal, specific attention is paid to the first  $n_y$  rows of Eq. (18). These rows satisfy

$$\begin{aligned} \mathbf{y}_M^+(t) &= \mathbf{C} \Omega_p(\tilde{\mathbf{A}}, [\mathcal{K} \quad \tilde{\mathbf{B}}]) \mathbf{Z}_{p,M}^- \\ &+ \mathbf{D} \mathbf{u}_M^+(t) + \mathbf{e}_M^+(t), \end{aligned} \quad (20)$$

whatever the used equation. Eq. (20) is nothing but the VARX model used in [13], [14]. From this standard model representation, using the idea of the proof of [27, Theorem 9.5], we can prove that, under Assumption 3,

$$\begin{aligned} \text{rank} \left( \lim_{M \rightarrow \infty} \frac{1}{M} \begin{bmatrix} \mathbf{Z}_{p,M}^- \\ \mathbf{u}_M^+(t) \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{p,M}^- \\ \mathbf{u}_M^+(t) \end{bmatrix}^\top \right) \\ = p(n_u + n_y) + n_u, \end{aligned} \quad (21)$$

while

- under open-loop conditions, with Assumption 1, we have

$$\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{e}_M^+(t) \begin{bmatrix} \mathbf{Z}_{p,M}^- \\ \mathbf{u}_M^+(t) \end{bmatrix}^\top = \mathbf{0}, \quad (22)$$

- under closed-loop conditions, with Assumption 1 and the constraint that  $\mathbf{D} = \mathbf{0}$ , we have

$$\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{e}_M^+(t) \left( \mathbf{Z}_{p,M}^- \right)^\top = \mathbf{0}. \quad (23)$$

Thanks to these results, the optimal prediction (in the least-squares sense) of  $\mathbf{y}_M^+(t)$  given the past input and output data as well as  $\mathbf{u}_M^+(t)$  is thus given by

$$\hat{\mathbf{y}}_M^+(t) = \mathbf{y}_M^+(t) / \begin{bmatrix} \mathbf{Z}_{p,M}^- \\ \mathbf{u}_M^+(t) \end{bmatrix} \quad (24)$$

where  $\bullet/\bullet$  stands for the oblique projection [25], *i.e.*, for two matrices  $\mathbf{N}$  and  $\mathbf{P}$  of appropriate dimensions,

$$\mathbf{N}/\mathbf{P} = \mathbf{N}\mathbf{P}^\dagger\mathbf{P}, \quad (25)$$

where  $\bullet^\dagger$  is the Moore Penrose pseudo inverse [32]. An optimal estimate in the least-squares sense of  $\mathbf{e}_M^+(t)$  is then obtained as follows

$$\hat{\mathbf{e}}_M^+(t) = \mathbf{y}_M^+(t) - \mathbf{y}_M^+(t) / \begin{bmatrix} \mathbf{Z}_{p,M}^- \\ \mathbf{u}_M^+(t) \end{bmatrix}. \quad (26)$$

Once  $\hat{\mathbf{e}}_M^+(t)$  is available, the system identification problem considered in this paper becomes, in a way, a deterministic system identification problem [11]. Thus, once the innovation term is estimated, several solutions suggested in the literature can be applied to get reliable estimates of the unknown matrices ( $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathcal{K}$ ).

In this paper, we focus on the extraction of the unknown matrices  $\mathbf{H}_f^{ol,e}$ ,  $\mathbf{H}_f^{ol,u}$ ,  $\mathbf{H}_f^{cl,u}$ ,  $\mathbf{H}_f^{cl,y}$ , respectively. The reasons why we focus on these matrices is twofold. First, it is clear from the definition  $\mathbf{H}_f^{ol,e}$ ,  $\mathbf{H}_f^{ol,u}$ ,  $\mathbf{H}_f^{cl,u}$ ,  $\mathbf{H}_f^{cl,y}$ , respectively, that they contain the Markov parameters of the unknown system. It is known from the 70's that standard realization techniques like the famous Kung's method [33] can be used to extract the matrices ( $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ ) from Markov parameters. Second, as a perspective for a future work, such an approach will allow us to incorporate prior information about the system by imposing prior knowledge on the impulse response (see [34] for more details about this idea).

##### B. Impulse response estimation (open loop case)

As explained previously, thanks to the VARX model (20), the innovation term involved in Eq. (18) is now known. Knowing<sup>1</sup> the Hankel matrices  $\mathbf{Z}_{p,M}^-$ ,  $\mathbf{E}_{f,M}^+$ ,  $\mathbf{U}_{f,M}^+$ ,  $\mathbf{Y}_{f,M}^+$

<sup>1</sup>In practice, we do not have access to these Hankel matrices but shifted version of them because the former step gives only access to  $\mathbf{e}(t)$ ,  $t = p, p+1, \dots$ . For the sake of simplification, we keep however the same notations afterwards.

a priori, with straightforward manipulations, the data equations gathered in Eq. (18a) can be written as the following regression equations

$$\mathbf{Y}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp = \begin{bmatrix} \mathbf{H}_f^{ol,u} & \mathbf{H}_f^{ol,e} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \\ \mathbf{E}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \end{bmatrix}, \quad (27)$$

where  $\mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp$  stands for the orthogonal projection onto  $\mathbf{Z}_{p,M}^-$ . Eq. (27) then defines a system of linear equations for the impulse response

$$\begin{bmatrix} \mathbf{D} & \mathbf{C}\mathbf{B} & \mathbf{C}\mathbf{A}\mathbf{B} & \dots & \mathbf{C}\mathbf{A}^{f-2}\mathbf{B} \end{bmatrix}, \quad (28a)$$

$$\begin{bmatrix} \mathbf{I}_{n_y} & \mathbf{C}\mathbf{K} & \mathbf{C}\mathbf{A}\mathbf{K} & \dots & \mathbf{C}\mathbf{A}^{f-2}\mathbf{K} \end{bmatrix}, \quad (28b)$$

of the system. In Section IV-D, we explain how to solve this structured linear problem.

For users familiar with the data-driven impulse response estimation technique developed in [35, Chapter 3], a slight modification of the impulse response computation algorithm called `w2h` in [35, page 78] can be easily suggested in order to take into account the estimated innovation and to give reliable estimates in a noisy framework. Indeed, instead of only considering  $\mathbf{u}$  and  $\mathbf{y}$  as the inputs of the function `w2h`, using the estimated innovation  $\hat{\mathbf{e}}$  as new input signals allows the user to have access to the impulse response coefficients given in Eq. (28). This technique, called *SSinnov2* in the following, will be used for the comparisons performed in Section V.

### C. Impulse response estimation (closed-loop case)

With straightforward manipulations, the data equations gathered in Eq. (18b) can be written as well as the following linear regression problem

$$\mathbf{E}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp = \begin{bmatrix} -\mathbf{H}_f^{cl,u} & \mathbf{I}_{n_y} - \mathbf{H}_f^{cl,y} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \\ \mathbf{Y}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \end{bmatrix}. \quad (29)$$

Again, we have access to a set of linear equations in terms now of

$$\begin{bmatrix} \mathbf{D} & \mathbf{C}\tilde{\mathbf{B}} & \mathbf{C}\tilde{\mathbf{A}}\mathbf{B} & \dots & \mathbf{C}\tilde{\mathbf{A}}^{f-2}\mathbf{B} \end{bmatrix}, \quad (30a)$$

$$\begin{bmatrix} \mathbf{0}_{n_y} & \mathbf{C}\mathbf{K} & \mathbf{C}\tilde{\mathbf{A}}\mathbf{K} & \dots & \mathbf{C}\tilde{\mathbf{A}}^{f-2}\mathbf{K} \end{bmatrix}, \quad (30b)$$

respectively.

### D. Structured Least-Squares Solution

In Eq. (27) and (29), it is important to notice that, besides sharing the same least-squares regression problem structure, the unknown matrices  $\mathbf{H}_f^{ol,u}$ ,  $\mathbf{H}_f^{ol,e}$ ,  $\mathbf{H}_f^{cl,u}$  and  $\mathbf{I}_{n_y} - \mathbf{H}_f^{cl,y}$ , respectively, all satisfy the same lower triangular Toeplitz structure. As shown, e.g., in [31], [36], [9], this structural constraint must be taken into account when standard 4SID techniques are used in order to ensure causality and consistency. In our case, the situation is a bit different and two cases can be considered.

1) When

- the system to identify belongs to the selected model structure, i.e., when the system's behavior can be described by Eq. (9),
- the innovation term is estimated optimally,

the over-determined sets of equations given in Eq. (27) and (29), respectively, are both consistent since, in this case,

$$\begin{aligned} & \left( \mathbf{Y}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \right)^\top \in \\ & \text{range} \left[ \left( \mathbf{U}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \right)^\top \quad \left( \mathbf{E}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \right)^\top \right], \end{aligned}$$

$$\begin{aligned} & \left( \mathbf{E}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \right)^\top \in \\ & \text{range} \left[ \left( \mathbf{U}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \right)^\top \quad \left( \mathbf{Y}_{f,M}^+ \mathbf{\Pi}_{\mathbf{Z}_{p,M}^-}^\perp \right)^\top \right], \end{aligned}$$

respectively. Then, because  $\begin{bmatrix} \mathbf{H}_f^{ol,u} & \mathbf{H}_f^{ol,e} \end{bmatrix}$  and  $\begin{bmatrix} -\mathbf{H}_f^{cl,u} & \mathbf{I}_{n_y} - \mathbf{H}_f^{cl,y} \end{bmatrix}$  are both full rank, the linear regression problems given in Eq. (27) and (29) have each a unique solution which satisfies the Toeplitz structural constraint.

- 2) In many practical cases, it is difficult to ensure that the system to identify belongs to the chosen LTI model class. In this case, the linear regression problems become inconsistent and structural constraints are necessary to guarantee that the estimates of  $\mathbf{H}_f^{ol,u}$ ,  $\mathbf{H}_f^{ol,e}$ ,  $\mathbf{H}_f^{cl,u}$  and  $\mathbf{I}_{n_y} - \mathbf{H}_f^{cl,y}$ , respectively, are lower triangular Toeplitz.

In order to guarantee the aforementioned structural constraint, we suggest a structured least-squares solution of the following generic problem

$$\mathbf{U}\mathbf{X}\mathbf{V} = \mathbf{W}, \quad (31)$$

where the matrices  $\mathbf{U}$ ,  $\mathbf{X}$ ,  $\mathbf{V}$  and  $\mathbf{W}$  are of appropriate dimensions,  $\mathbf{U}$ ,  $\mathbf{X}$  and  $\mathbf{V}$  are assumed to be known while the matrix  $\mathbf{X}$  is constrained to be a lower triangular Toeplitz matrix. This structured least-squares problem appears four times in Eq. (27) and Eq. (29). Such a constrained least-squares problem can be solved as follows. Given a generic lower triangular Toeplitz matrix  $\mathbf{X} \in \mathbb{R}^{xi \times yi}$  of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_0 & & & & \\ \mathbf{x}_1 & \mathbf{x}_0 & & & \\ \vdots & & \ddots & & \\ \mathbf{x}_{i-1} & \dots & \mathbf{x}_1 & \mathbf{x}_0 & \end{bmatrix}, \quad (32)$$

with individual blocks having dimensions  $\mathbf{x}_k \in \mathbb{R}^{x \times y}$ ,  $k = 0, \dots, i-1$ , we can prove that

$$\mathbf{X} = \sum_{k=0}^{i-1} \mathbf{F}_k \otimes \mathbf{x}_k, \quad (33)$$

where the basis matrices  $\mathbf{F}_k$  are defined as follows

$$\mathbf{F}_0 = \mathbf{I}_i, \quad (34a)$$

$$\mathbf{F}_1 = \begin{bmatrix} \mathbf{0}_{1 \times i-1} & 0 \\ \mathbf{I}_{i-1} & \mathbf{0}_{i-1 \times 1} \end{bmatrix}, \quad (34b)$$

$$\mathbf{F}_2 = \begin{bmatrix} \mathbf{0}_{2 \times i-1} & \mathbf{0}_2 \\ \mathbf{I}_{i-2} & \mathbf{0}_{i-2 \times 2} \end{bmatrix}, \quad (34c)$$

$$\vdots \quad (34d)$$

$$\mathbf{F}_{i-1} = \begin{bmatrix} \mathbf{0}_{i-1 \times 1} & \mathbf{0}_{i-1} \\ 1 & \mathbf{0}_{1 \times i-1} \end{bmatrix}, \quad (34e)$$

while  $\otimes$  stands for the Kronecker product [32]. Going back to Eq. (31), using the fact that  $\text{vec}(\mathbf{U}\mathbf{X}\mathbf{V}) = (\mathbf{V}^\top \otimes \mathbf{U}) \text{vec}(\mathbf{X})$  [32], we get

$$(\mathbf{V}^\top \otimes \mathbf{U}) \text{vec} \left( \sum_{k=0}^{i-1} \mathbf{F}_k \otimes \mathbf{x}_k \right) = \text{vec}(\mathbf{W}). \quad (35)$$

Focusing now on  $\text{vec} \left( \sum_{k=0}^{i-1} \mathbf{F}_k \otimes \mathbf{x}_k \right)$ , it can be proved that

$$\text{vec}(\mathbf{X}) = \sum_{k=0}^{i-1} ((\mathbf{I}_i \otimes \mathbf{K}_{y_i}) (\text{vec}(\mathbf{F}_k) \otimes \mathbf{I}_y) \otimes \mathbf{I}_x) \times \text{vec}(\mathbf{x}_k), \quad (36)$$

where  $\mathbf{K}_{y_i} \in \mathbb{R}^{y_i \times y_i}$  is a commutator matrix (see [37] for details). Using standard manipulations, Eq. (35) can be written as follows

$$(\mathbf{V}^\top \otimes \mathbf{U}) \mathcal{F} \begin{bmatrix} \text{vec}(\mathbf{x}_0) \\ \vdots \\ \text{vec}(\mathbf{x}_{i-1}) \end{bmatrix} = \text{vec}(\mathbf{W}), \quad (37)$$

where

$$\mathcal{F} = [\mathcal{F}_0 \quad \cdots \quad \mathcal{F}_{i-1}] \quad (38)$$

with, for  $k = 0, \dots, i-1$ ,

$$\mathcal{F}_k = ((\mathbf{I}_i \otimes \mathbf{K}_{y_i}) (\text{vec}(\mathbf{F}_k) \otimes \mathbf{I}_y) \otimes \mathbf{I}_x). \quad (39)$$

The determination of  $\mathbf{X}$  is finally possible via the calculation of the pseudo inverse of  $(\mathbf{V}^\top \otimes \mathbf{U}) \mathcal{F}$  followed by the re-assembling of the elements  $\{\mathbf{x}_0, \dots, \mathbf{x}_{i-1}\}$  into the lower triangular Toeplitz matrix  $\mathbf{X}$ .

## V. EXAMPLES

Two simulation examples under open-loop and closed-loop conditions, respectively, are now considered. In each case,

- a Monte Carlo simulation is carried out with 100 different realizations of the the zero-mean white Gaussian noise  $\mathbf{e}$ ,
- the techniques described in Sub-Sections IV-B-IV-C-IV-D (called SSinnov1 afterwards) are compared with the SSinnov2 method, *i.e.*, a modified w2h algorithm [35] (which uses the innovation estimate as a new input), the SSARX algorithm [14] as well as the famous N4SID algorithm (implemented in MATLAB (function `n4sid` called with the default parameters)).

In order to quantify the performance of the algorithms compared hereafter, we use the following fit measurements.

$$\text{BFT} = 100 \times \max \left( 0, 1 - \frac{\|\mathbf{y} - \hat{\mathbf{y}}\|_2}{\|\mathbf{y} - \text{mean}(\mathbf{y})\|_2} \right), \quad (40a)$$

$$\text{VAF} = 100 \times \max \left( 0, 1 - \frac{\text{var}(\mathbf{y} - \hat{\mathbf{y}})}{\text{var}(\mathbf{y})} \right), \quad (40b)$$

$$\mathbf{e}_{\mathbf{G}} = \frac{\|\mathbf{G} - \hat{\mathbf{G}}\|_2}{\|\mathbf{G}\|_2}, \quad (40c)$$

$$\mathbf{e}_{\mathbf{h}} = \frac{\|\mathbf{h} - \hat{\mathbf{h}}\|_2}{\|\mathbf{h}\|_2}, \quad (40d)$$

where  $\mathbf{y}$  is the system's output,  $\hat{\mathbf{y}}$  is the model's output,  $\mathbf{h}$  is the system's impulse response,  $\hat{\mathbf{h}}$  is the model's impulse response,  $\mathbf{G}$  is the system's representation while  $\hat{\mathbf{G}}$  is the model's representation.

### A. Open-loop case

We consider the discrete-time system (see [25, Section 4.4.5] for details) with

$$\begin{aligned} A &= 0.749, & B &= 1.8805, & C &= 0.8725, \\ D &= -2.0895, & K &= -0.0580, & R &= 6.705, \end{aligned}$$

where  $\mathbb{E}[e^2(t)] = R$ . The sampling period is chosen equal to 1 s. For this specific system, the state-space representation of which perfectly matches the innovation model given in Eq. (9), we compare the algorithms described in Sub-Section IV-B (denoted hereafter by SSinnov1 and SSinnov2) with N4SID and SSARX. For this simulation example, we consider 4000 samples for each realizations with  $u$  a zero mean, white, random process drawn from the standard normal distribution with a unit standard deviation.

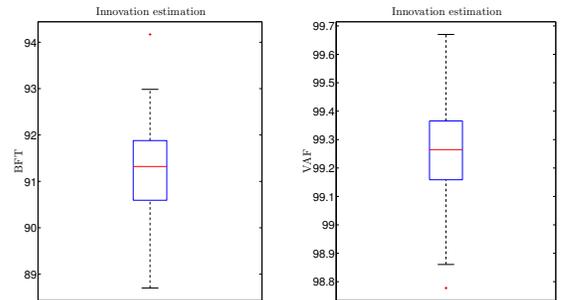


Fig. 1: Comparison, for each realization, of the estimated and randomly generated innovation signals. BFT on the left-hand side, VAF on the right-hand side.

First, Figure 1 illustrates the efficiency of the technique described in Sub-Section IV-A to estimate the innovation term. We see from these figures that the reconstruction of the innovation term is reliable for this open-loop system. Once this innovation term is available, the techniques SSinnov1 and SSinnov2 can be compared to N4SID and SSARX, respectively. This comparison is done by computing, for each realization, the performance indices  $\mathbf{e}_{\mathbf{G}}$  and  $\mathbf{e}_{\mathbf{h}}$ , *i.e.*,

by quantifying the capabilities of the techniques involved in this comparison to mimic the impulse response of the system as well as its input-output representation. Figure 2 shows that (i) all the techniques perform quite well, (ii) the best results are obtained with SSARX, (iii) SSinnov1 is as efficient as SSARX to mimic the global behavior of the system while (iv) SSinnov2 and N4SID perform similarly.

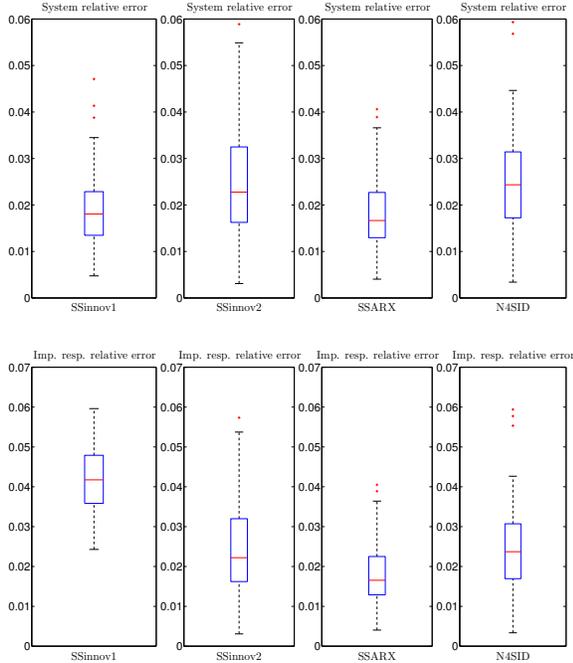


Fig. 2: Relative errors  $e_G$  and  $e_h$  for SSinnov1, SSinnov2, SSARX and N4SID, respectively.

### B. Closed-loop case

We use now the simulation example suggested in the System Identification Toolbox of MATLAB, *i.e.*, a second-order system corrupted by white noise and generated as follows

```

N = 1000;
K = 0.5;
r = randn(N, 1);
z = zeros(N, 1);
u = zeros(N, 1);
y = zeros(N, 1);
e = randn(N, 1);
v = filter([1 0.5], [1 1.5 0.7], e);
for k = 3:N
    u(k-1) = -K*y(k-1) + r(k);
    z(k) = 1.5*z(k-1) - 0.7*z(k-2) + ...
        u(k-1) + 0.5*u(k-2);
    y(k) = z(k) + 0.8*v(k);
end

```

Again, we compare SSinnov1 (implemented using the steps described in Sub-Sections IV-C-IV-D) and SSinnov2 with N4SID and SSARX but, in this case, by comparing the capabilities of the models to simulate the system's output. The system's description does not satisfy the innovation

model given in Eq. (9) anymore. The FIT and VAF values gathered in Figure 3 show that (i) the best performance is obtained with SSARX, (ii) SSinnov1 is right on SSARX's heels as far as the reconstruction of the output signal is concerned, (iii) SSinnov2 is less efficient for this simulation example.

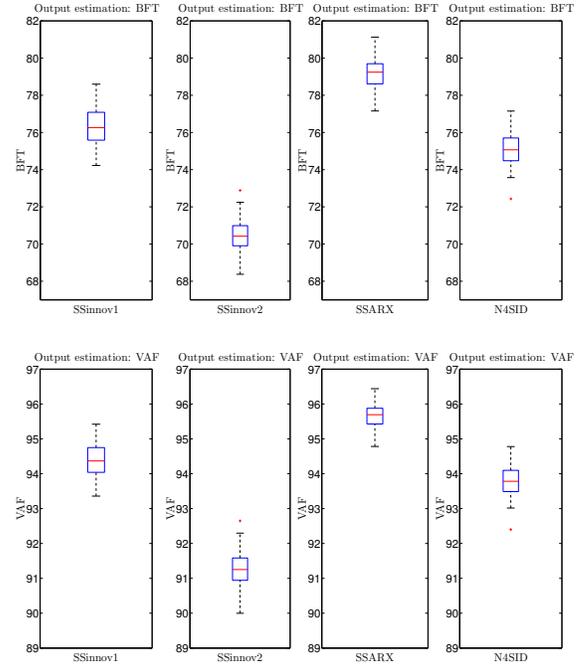


Fig. 3: Comparison, for each realization, of the capabilities of the models estimated with SSinnov1, SSinnov2, SSARX and N4SID, respectively, to reconstruct the system's output. BFT at the top, VAF at the bottom.

## VI. CONCLUSIONS AND FUTURE WORKS

In order to bypass usual difficulties occurring when noisy data are handled, a two-step 4SID procedure is suggested in this paper. Starting from the estimation of the innovation term using a VARX model, least-squares-based solutions are developed to estimate the Markov parameters of the system to identify. The impulse response realization problem is a well-known problem for which efficient techniques are now available. In this paper, we focus on solutions (i) working under open-loop and closed-loop conditions, respectively, (ii) dealing with Toeplitz matrices usually encountered in subspace-based system identification. The future works will consist in modifying these algorithms in order to take into account prior knowledge described as constraints on the impulse response, *e.g.*, steady-state gain, overshoot, and rise time, via equality or inequality constraints.

## REFERENCES

- [1] M. Verhaegen, A. Varga, and G. Grübel, "Some experience with the MOESP class of subspace model identification methods in identifying the BO105 helicopter," tech. rep., German Aerospace Research Establishment, Laboratory for Flight Systems Dynamics, Wessling, Germany, 1994.

- [2] W. Favoreel, B. De Moor, and P. Van Overschee, "Subspace state space system identification for industrial processes," *Journal of Process Control*, vol. 10, pp. 149–155, 2000.
- [3] K. De Cock, G. Mercère, and B. De Moor, "Recursive subspace identification for in flight modal analysis of airplanes," in *Proceedings of the International Conference on Noise and Vibration Engineering*, (Leuven, Belgium), September 2006.
- [4] M. Viberg, "Subspace-based methods for the identification of linear time-invariant systems," *Automatica*, vol. 31, pp. 1835–1851, 1995.
- [5] D. Bauer, "Asymptotic properties of subspace estimators," *Automatica*, vol. 41, pp. 359–376, 2005.
- [6] J. van Wingerden and M. Verhaegen, "Subspace identification of bilinear and LPV systems for open- and closed-loop data," *Automatica*, vol. 45, pp. 372–381, 2009.
- [7] M. Bergamasco and M. Lovera, "Continuous-time predictor-based subspace identification using Laguerre filter," *IET Control Theory and Applications*, vol. 5, pp. 856–867, 2011.
- [8] L. Ljung, *System identification. Theory for the user*. Upper Saddle River: Prentice Hall, 2nd ed., 1999.
- [9] S. Qin, "An overview of subspace identification," *Computers and Chemical Engineering*, vol. 30, pp. 1502–1513, 2006.
- [10] L. Ljung and T. McKelvey, "Subspace identification from closed loop data," *Signal Processing*, vol. 52, pp. 209–215, 1996.
- [11] D. Di Ruscio, "Closed and open loop subspace system identification of the kalman filter," *Modeling, Identification and Control*, vol. 30, pp. 71–86, 2009.
- [12] A. Chiuso, "The role of vector autoregressive modeling in predictor-based subspace identification," *Automatica*, vol. 43, pp. 1034–1048, 2007.
- [13] M. Jansson, "Subspace identification and ARX modeling," in *Proceedings of the IFAC Symposium on System Identification*, (Rotterdam, The Netherlands), August 2003.
- [14] M. Jansson, "A new subspace identification method for open and closed loop data," in *Proceedings of the IFAC World Congress*, (Prague, Czech Republic), July 2005.
- [15] C. Lawson and R. Hanson, *Solving least squares problems*. Society for Industrial Mathematics, 1995.
- [16] I. Markovsky and P. Rapisarda, "Data-driven simulation and control," *International Journal of Control*, vol. 81, pp. 1946–1959, 2008.
- [17] J. Willems, P. Rapisarda, I. Markovsky, and B. De Moor, "A note on persistency of excitation," *Systems and Control Letters*, vol. 54, pp. 325–329, 2005.
- [18] M. Gevers and B. Anderson, "Representation of jointly stationary feedback free processes," *International Journal of Control*, vol. 33, pp. 777–809, 1981.
- [19] M. Gevers and B. Anderson, "On jointly stationary feedback free stochastic processes," *IEEE Transactions on Automatic Control*, vol. 27, pp. 431–436, 1982.
- [20] W. Lin, S. Qin, and L. Ljung, "A framework for closed loop subspace identification with innovation estimation," tech. rep., Texas Wisconsin Modeling and Control Consortium, Wisconsin, USA, 2004.
- [21] A. Chiuso, "On the asymptotic properties of closed-loop CCA-type subspace algorithms: equivalence results and the role of the future horizon," *IEEE Transactions on Automatic Control*, vol. 55, pp. 634–649, 2010.
- [22] G. Pannocchia and M. Calosi, "A predictor form PARSIMonious algorithm for closed-loop subspace identification," *Journal of Process Control*, vol. 20, pp. 517–524, 2010.
- [23] B. Wahlberg and M. Jansson, "4SID linear regression," in *Proceedings of the Conference on Decision and Control*, (Lake Buena Vista, Florida, USA), December 1994.
- [24] M. Jansson and B. Wahlberg, "A linear regression approach to state space subspace system identification," *Signal Processing*, vol. 52, pp. 103–129, 1996.
- [25] P. Van Overschee and B. De Moor, *Subspace identification for linear systems. Theory, implementation, applications*. Kluwer Academic Publishers, 1996.
- [26] T. Katayama, *Subspace methods for system identification*. Springer Verlag, 2005.
- [27] M. Verhaegen and V. Verdult, *Filtering and system identification: a least squares approach*. Cambridge University Press, 2007.
- [28] K. Peternell, *Identification of linear dynamic systems by subspace and realization-based algorithms*. PhD thesis, Technical University of Vienna, Vienna, Austria, 1995.
- [29] M. Jansson, *On subspace methods in system identification and sensor array signal processing*. PhD thesis, Royal Institute of Technology, Stockholm, Sweden, 1997.
- [30] D. Bauer, *Some asymptotic theory for the estimation of linear systems using maximum likelihood methods or subspace algorithms*. PhD thesis, Vienna University of Technology, Vienna, Austria, 1998.
- [31] K. Peternell, W. Sherrer, and M. Deistler, "Statistical analysis of novel subspace identification methods," *Signal Processing*, vol. 52, pp. 161–177, 1996.
- [32] R. Horn and C. Johnson, *Matrix analysis*. Cambridge University Press, 1990.
- [33] S. Kung, "A new identification and model reduction algorithm via singular value decomposition," in *Proceedings of the Asilomar Conference on Circuits, Systems and Computer*, (Pacific Grove, California, USA), November 1978.
- [34] I. Markovsky and G. Mercère, "Subspace identification with constraints on the impulse response." Submitted to *International Journal of Control*.
- [35] I. Markovsky, *Low rank approximation: algorithms, implementation, applications*. Communications and Control Engineering, Springer, 2012.
- [36] S. Qin and L. Ljung, "Closed loop subspace identification with innovation estimation," in *Proceedings of the IFAC Symposium on System Identification*, (Rotterdam, The Netherlands), August 2003.
- [37] J. Magnus and H. Neudecker, *Matrix differential calculus with applications in statistics and econometrics*. Wiley, 1999.