Model order estimation based on a method for computing distance to uncontrollability

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Abstract

A classical tool for spurious poles detection in modal analysis is the stabilization diagram. It is widely used by practitioners because of its simplicity and intuitive appeal. Despite of its popularity the stabilization diagram requires subjective human judgment and lacks theoretical justification. In this paper, we propose an new approach to spurious poles detection that 1) has theoretical justification and 2) does not make unverifiable assumptions about the data generating system, apart form the basic one that it is linear time-invariant. The method is based on a quantitative measure of the distance of the identified model to a system with an eliminated spurious pole. Small distance indicates that the eliminated pole does not degrade the model quality. The procedure can be iterated for elimination of multiple spurious poles. An increase of the distance indicates that all spurious poles are eliminated from the model.

1 Introduction

An important problem in modal analysis is the estimation of the number of poles n of the system. Underestimating n ($\widehat{n} < n$) results in $\widehat{n} - n$ of the system poles not being included into the model. Overestimating n ($\widehat{n} > n$) results in $n - \widehat{n}$, so called *spurious poles*, appearing in the model. The spurious poles are an artifact of the disturbances and measurement noise. They depend also on the estimation method being used.

In vibration analysis, stabilization diagrams [1] are used for separation of the "true system's poles", called *physical poles*, from the spurious poles. A stabilization diagram shows the model poles for increasing orders of the model. The separation principle is that the physical poles tend to "stabilize" (*i.e.*, when the order is increased they do not change) while the spurious poles do not stabilize (*i.e.*, they change randomly). This behavior is empirically observed. However, it is not supported theoretically. In general, when the data is noisy the physical poles of the statistically optimal model of order $\widehat{n} > n$ are not a subset of the poles of the statistically optimal model of order $\widehat{n} = n$.

There are alternatives to the stabilization diagrams for finding the value of n and estimating the n physical poles. As discussed in [2], the problem is closely related to the problems of the model order selection in identification and model order reduction in systems theory. The order selection problem refers to inferring the value of n from data. Classical methods, used in system identification, are the Akaike information criterion [3], the minimum description length [4], the singular values of the Hankel matrix, and their numerous variations [5]. The order reduction problem refers to approximation of a given model of order n by a model of lower order [6]. Classical techniques, used in control theory, are the balanced reduction [7], Hankel model reduction [8], and H_2/H_∞ model reduction [9]. Balanced model reduction has been used for spurious poles detection in [10]. Model reduction is also closely related to the Pade approximation and rational approximation problems in mathematics [11].

In this paper we present a new procedure for order estimation motivated by recent results [12] on the computation of the distance of a given model to the set of uncontrollable systems (the distance to uncontrollability problem) [13, 14]. For a single-input single-output system, it is shown that computing the distance to uncontrollability is equivalent to computing an approximate common factor of two polynomials. Common factor computation, in turn, is an old problem in mathematics related to classical results such as Euclid's method and the Sylvester resultant. Nowadays, the problem of computing optimal in the sense of perturbation of the polynomials' coefficients approximate factors is a major research topic in computer algebra, see, *e.g.*, [15].

The link between distance to uncontrollability and order selection is the fact that the identified model is nonminimal (or, equivalently, has at least one spurious pole) if and only if it is uncontrollable or, equivalently, the numerator and denominator of a transfer function representation of the model have a nontrivial common factor. In the presence of disturbances and measurement noise, the model is not exact, which results in the transfer function coefficients being perturbed. In this case, the detection of a common factor is a nontrivial problem. The method for approximate common factor estimation proposed in [12] is based on integration of ordinary differential equations. It is computationally fast and empirical results show that it is more robust than existing methods based on classical optimization methods [16, 17, 18]. Here we use the method of [12] for order estimation and show its performance examples.

The paper is organized as follows. Section 2 defines the problem and the notation used in the rest of the paper. Section 3 reviews a method based on singular value decomposition of the Hankel matrix constructed from the data. This method is widely used in system identification and signal processing and is our reference method. Section 4 describes the newly proposed method based on distance to uncontrollability. Section 5 shows numerical results comparing the proposed method with the reference method.

2 Problem formulation

The problem considered is defined as follows:

> Given an input/output trajectory $w = (u, y)$ of a dynamical system $\mathscr{B}$, estimate the order of $\mathscr{B}$.

The data is a finite-length sampled trajectory $w = \big(w(1), \dots, w(T)\big)$. The order n of a linear time-invariant system $\mathscr{B}$ is equal to the number of the (physical) poles of $\mathscr{B}$. In what follows, we assume that the data generating system is linear time-invariant. Moreover, since the method of [12] is developed for single-input single-output systems, we also limit the scope of the paper to the single-input single-output case.

A key notion in the solution of the order estimation problem turns out to be the persistency of excitation [19, Definition 8.14]. Define the Hankel matrix with t block rows, composed of the sequence $w \in (\mathbb{R}^2)^T$

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-t+1) \\ w(2) & w(3) & \cdots & w(T-t+2) \\ w(3) & w(4) & \cdots & w(T-t+3) \\ \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & \cdots & w(T) \end{bmatrix}.$$

The time series $u = \big(u(1),\ldots,u(T)\big)$ is persistently exciting of order L if the Hankel matrix $\mathscr{H}_L(u)$ is of full row rank. The fact that u is persistency exciting of order L implies that there is no exact linear time-invariant system of order L (or less) that fits exactly u.

3 A classical method for order estimation

A classical way of order estimation, used in subspace system identification and signal processing, is based on the decay of the singular values of the Hankel matrix $\mathscr{H}_L(w)$, where the parameter L is selected so that the matrix $\mathscr{H}_L(w)$ is nearly square, *i.e.*, $L = \lceil T/2 \rceil$. In case of exact output only data (or impulse response data) and assuming that the number of rows as well as the number of columns of $\mathscr{H}_L(w)$ are bigger than the order n of the system, it can be shown that $\operatorname{rank}\big(\mathscr{H}_L(w)\big) = n$, *i.e.*, under the specified assumption the number of nonzero singular values is equal to system's order. When the data is noisy but the noise variance is sufficiently small, the singular values have a drop at the index corresponding to the system's order, *i.e.*,

$$\sigma_n \gg \sigma_{n+1}. \tag{1}$$

In the case of a single input single output data, assuming in addition that the input is persistently exciting

$$\operatorname{rank}\big(\mathscr{H}_L(w)\big) = L + n. \tag{2}$$

Thus the rank deficiency of the Hankel matrix again allows us to determine the order using numerical rank computation methods, such as the singular value decomposition.

4 The proposed method

The solution procedure proposed in this paper is based on the following theoretical results: assuming that

1. the data is generated by a controllable linear time-invariant system of order n, and
2. the input component u is persistently exciting of order $2n+1$,

an exact model $\widehat{\mathscr{B}}$ of order $\widehat{n} > n$ for the data w is uncontrollable. Moreover, there is no exact model $\widehat{\mathscr{B}}$ of order $\widehat{n} < n$ for the data w.

The result shows that under assumptions 1 and 2, the least complicated (*i.e.*, smallest order) exact model for w has order equal to n. Therefore, under assumptions 1 and 2, the problem of finding n from the data w is equivalent to the problem of finding the least complicated exact model for w.

Assumption 1, however, is unrealistic. Even if the data generating system is linear time-invariant, it is affected by unobserved inputs (disturbances) and the trajectory w is observed with measurement noise, *i.e.*,

$$w = \bar{w} + \widetilde{w}, \tag{3}$$

where $\bar{w}$ is the true trajectory and $\widetilde{w}$ is the measurement noise. In case of disturbances and measurement noise present, an exact model for the data is generically has order $\widehat{n} = T$. In this case, an approximate model of bounded order $\widehat{n} < T$ is needed. If the noise variance is small compared with the variance of the true signal, an approximate model $\widehat{\mathscr{B}}$ of order $\widehat{n} > n$ for the data w is still "close to uncontrollable".

The idea of the proposed procedure is to monitor the distance to controllability for increasing values of the model order $\widehat{n}$. Let $\widehat{\mathscr{B}}_{\widehat{n}}$ be an approximate model of order $\widehat{n}$ for the data w and let $d_{\widehat{n}}$ be the distance of $\widehat{\mathscr{B}}_{\widehat{n}}$ to the set of uncontrollable systems. For small noise variance, $d_{\widehat{n}}$ drops at $\widehat{n} = n$, *i.e.*, the difference $\Delta d_{\widehat{n}} = d_{\widehat{n}} - d_{\widehat{n}+1}$ has a peak at $\widehat{n} = n$. The order estimation procedure is:

- given data $w = (u, y)$ and a threshold ε
- for $\widehat{n} = 1, 2, \ldots$ do
  1. using the data w, identify an approximate model $\widehat{\mathscr{B}}_n$ of order $\widehat{n}$,
  2. compute the distance $d_{\widehat{n}}$ of $\widehat{\mathscr{B}}_{\widehat{n}}$ to the set of uncontrollable system,
  3. if $d_{\widehat{n}} < \varepsilon$, then stop.
- output model $\widehat{\mathscr{B}} = \widehat{\mathscr{B}}_{\widehat{n}}$ for w of order $n = \widehat{n}$

Any identification method can be used on step 1. In the numerical examples presented in Section 5, we use Kung's method [20] in the case of autonomous system and the N4SID subspace identification method [21] (implemented in the System Identification Toolbox of Matlab [22]) in the case of an input-output system. The distances $d_{\widehat{n}}$ of the identified model to uncontrollability in step 2 is the smallest (measured in 2-norm) perturbation of the model parameters that renders the system uncontrollable. The method for its computation is based on integration of ordinary differential equations (see [12]) and is available from:

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http://homepages.vub.ac.be/~imarkovs/software/unctr.tar
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5 Numerical example

First, we test the order estimation method based on the distance to uncontrollability on a simulation exam and compare the results with the ones of the classical order estimation method based on the singular value decomposition of the Hankel matrix. Then, we show the performance of the method on real-life data of single-input single-output systems from the database for system identification DAISY [23].

5.1 Test on a simulation example

Consider the measurement errors setup (3). The true trajectory $\bar{w}$ is an impulse response of a linear time-invariant system of order $n = 4$, see Figure 1.

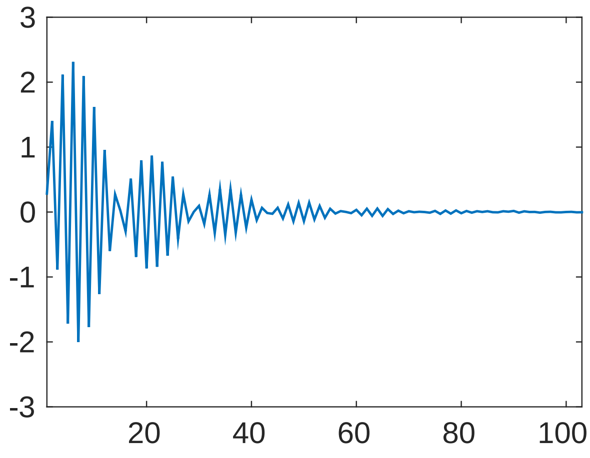

Figure 1: Impulse response of the system in the simulation example.

We compare the order estimation methods based on the singular value decomposition and the distance to uncontrollability. For the model identification we use Kung's method [20]. The normalized distance $d_{\text{norm}}(\widehat{n})$ of the identified models with order $\widehat{n} = 1, \dots, 7$ to the set of uncontrollable systems is shown in Figure 2, left. The normalization is $d_{\text{norm}}(\widehat{n}) := d(\widehat{n})/d(1)$, where d is the distance to uncontrollability. The normalized first 7 singular values s_{norm} of the Hankel matrix $\mathscr{H}_{50}(h)$ are shown in Figure 2, right. The normalization is $s_{\text{norm}}(\widehat{n}) := s(\widehat{n})/s(1)$, where $s(1), \dots, s(50)$ are the singular values of $\mathscr{H}_{51}(h)$.

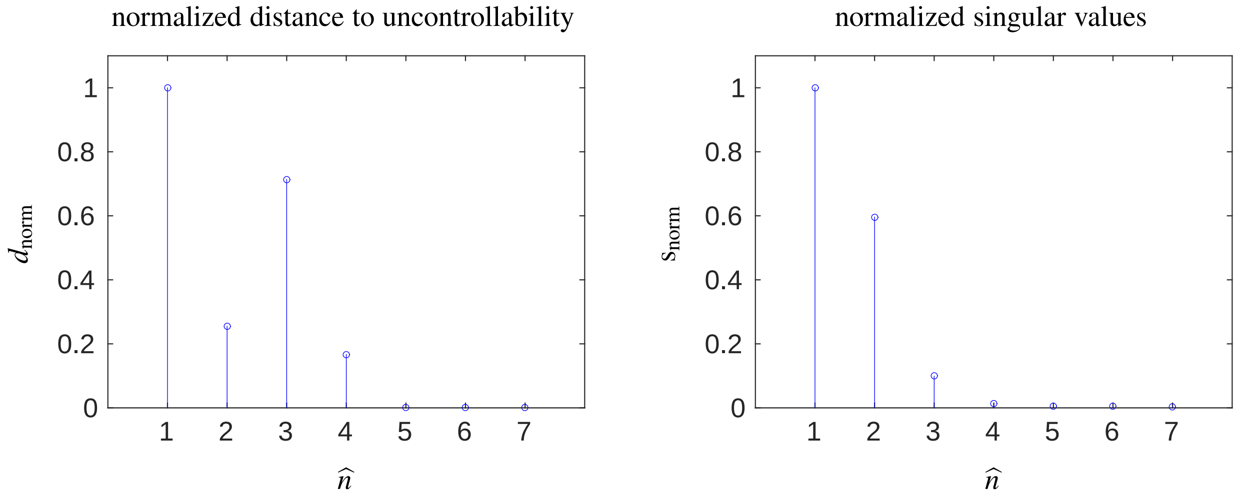

Figure 2: Normalized distance to uncontrollability d_{norm} and normalized singular values s_{norm}.

As an order selection criterion we use

$$\arg\max_{\widehat{n}} \big(\Delta d_{\text{norm}}(\widehat{n})\big), \tag{4}$$

where $\Delta d_{\text{norm}}(\widehat{n})$ is the relative difference $\big(d(\widehat{n}) - d(\widehat{n}-1)\big)/d(\widehat{n})$, normalized, so that $\max_{\widehat{n}} \Delta d_{\text{norm}}(\widehat{n}) = 1$. Similarly, the order selection criterion using the singular value decomposition is maximization of Δs_{norm} — the relative difference of the singular values $\big(s(\widehat{n}) - s(\widehat{-}1)\big)/s(\widehat{n})$, normalized so that its maximum over $\widehat{n}$ is equal to 1. Figure 3 shows Δd_{norm} (left) and Δs_{norm} (right).

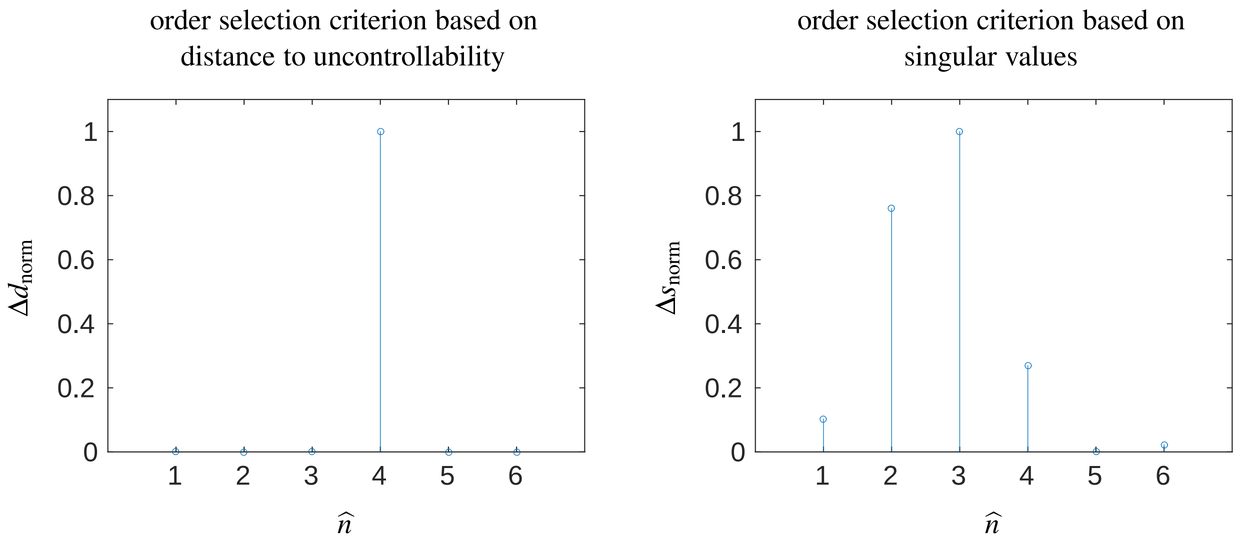

Figure 3: Order selection criteria based on the distance to uncontrollability (left) and singular values (right).

In the example, the maximum value of Δd_{norm} is at the correct model order $\widehat{n} = n = 4$. In the next section, we apply the criterion (4) on data from real-life examples where there true model does not exist.

5.2 Example with a real-life data

In this section, we perform numerical tests with single-input single-output data sets from the database for system identification DAISY [23], see Table 1.

Data set name	File name	$\widehat{n}$
Heating system	`heating_system`	1–3
Data of a laboratory setup acting like a hair dryer	`dryer`	4
Data of the ball-and-beam setup in SISTA	`ballbeam`	1, 5
Wing flutter data	`flutter`	5
Data from a flexible robot arm	`robot_arm`	1–2
Liquid-saturated steam heat exchanger	`exchanger`	1

Table 1: Data sets from the database for system identification DAISY [23], used in the simulation examples.

- *Heating system:* The input drives a 300 Watt Halogen lamp, suspended above a thin steel plate. The output is a thermocouple measurement taken from the back of the plate.
- *Hair dryer:* This data is obtain from a laboratory setup acting like a hair dryer. Air is fanned through a tube and heated at the inlet. The air temperature is measured by a thermocouple at the output. The input is the voltage over the heating device (a mesh of resistor wires).
- *Ball-beam:* The data is of a the ball and beam practicum at the research group ESAT-SISTA of the K.U.Leuven, Belgium. The input is the angle of the beam. The output is the position of the ball.
- *Flutter:* Due to industrial secrecy agreements, details about this dataset are not revealed.
- *Robot arm:* The arm is installed on an electrical motor. The transfer function from the measured reaction torque of the structure on the ground to the acceleration of the flexible arm is modeled. The applied input is a periodic sine sweep. The input is reaction torque of the structure. The output is acceleration of the flexible arm.
- *Exchanger:* The process is a liquid-satured steam heat exchanger, where water is heated by pressurized saturated steam through a copper tube. The output variable is the outlet liquid temperature. The input variables are the liquid flow rate, the steam temperature, and the inlet liquid temperature. In this experiment the steam temperature and the inlet liquid temperature are kept constant to their nominal values. The heat exchanger process is a significant benchmark for nonlinear control design purposes, since it is characterized by a non minimum phase behavior. The input is the liquid flow rate. The output is the outlet liquid temperature.

The plots of the criterion Δd_{norm} for the considered benchmarks are shown in Figure 4. Then, applying the order selection rule (4), we obtain the model order estimates shown in the last column of Table 1.

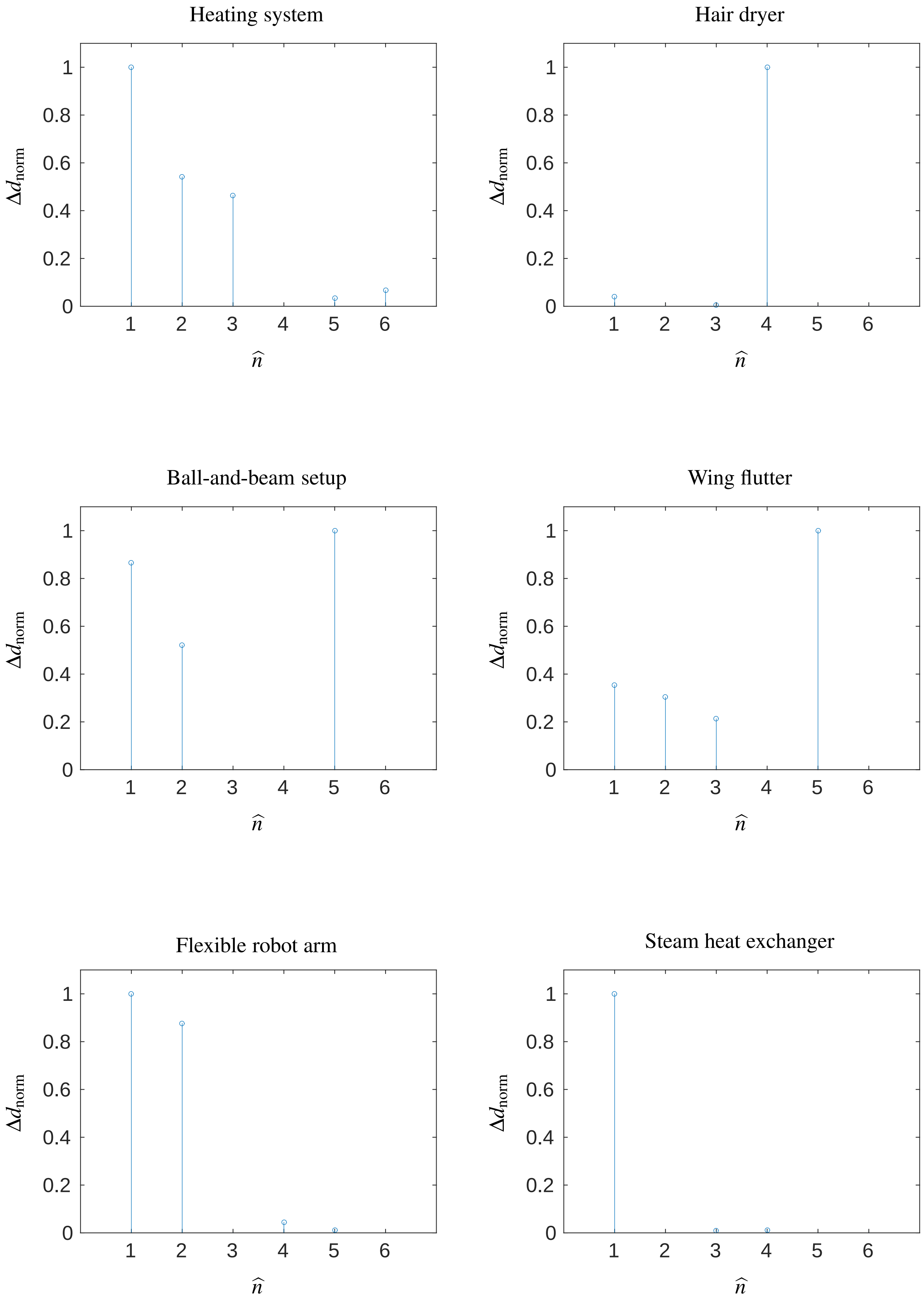

Figure 4: Order selection criteria based on the distance to uncontrollability for the benchmarks of DAISY.

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