On the persistency of excitation

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Abstract

The result of J.C. Willems et al. "A note on persistency of excitation", System & Control Letters, 2005 gives identifiability conditions for system identification as well as data-driven representations for data-driven control. The existing proofs however are proofs by contradiction and do not give insight into the assumptions of controllability and persistency of excitation of the input. Moreover, the existing proofs do not clarify how conservative the assumptions are. We provide an alternative constructive proof for the single-input case. It is shown that persistency of excitation of order more than the time horizon is needed in nongeneric cases, corresponding to special initial conditions. The special initial conditions are explicitly characterized in terms of the solution of a Sylvester equation. Another contribution of the paper is a representation of a scalar persistently exciting input of a finite order as an output of an autonomous linear time-invariant system.

Key words: behavioral approach; exact identification; identifiability; persistency of excitation; input design.

1 Introduction

A class of subspace-type system identification and data-driven control methods for discrete-time linear time-invariant systems originate from a result that became known as the *fundamental lemma* [13]. The fundamental lemma gives identifiability conditions, *i.e.*, conditions under which the data-generating system can be recovered back from the data. At the same time, the fundamental lemma gives a nonparametric representation of the finite-horizon behavior of the data-generating system. This nonparametric representation is completely specified by a single trajectory of the system. For an overview of the methods based on the fundamental lemma, we refer the reader to [5].

The data-driven nonparametric representation of the system is given by a Hankel matrix constructed from the data. Under the conditions of the fundamental lemma, the image of the Hankel matrix equals the set of all finite-length trajectories of the data generating system. The conditions of the fundamental lemma restrict the input signal and the data-generating system, so that an observed input/output trajectory of the system completely reveals the system's dynamics. The key condition of the fundamental lemma is that the input signal be persistently exciting of order equal to the length the finite horizon plus the order of the system.

Although the fundamental lemma is widely used and has numerous generalizations, see, *e.g.*, [8,3,9,10], there are important open problems related to it. The original proof [13] as well as the subsequent proof of [9] are by contradiction and do not give insight into the need and the meaning of the persistency of excitation condition. Indeed, the conditions of the fundamental lemma are sufficient only and it is unknown how conservative they are. Our main contributions are:

(1) an alternative constructive proof of the fundamental lemma in the single-input case that shows the nonconservatism of the fundamental lemma,
(2) explicit characterization of the cases, in which persistency of excitation of order more than the horizon's length is needed, and
(3) a representation of a scalar persistently exciting input of a specified order as an output of an autonomous linear time-invariant system.

The third contribution can be used for design of informative experiments with additional constraints on the input.

⋆ This paper was presented at the MTNS meeting 2022. Corresponding author I. Markovsky.

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2 Background and problem statement

We use the behavioral language, where a dynamical system $\mathscr{B}$ is a set of trajectories [12,5]. In this paper, we consider deterministic discrete-time linear time-invariant dynamical systems. Let q be the number of variables. A finite T-sample long trajectory of the system $\mathscr{B}$ is a time series

$$w_{\rm d} = \big(w_{\rm d}(1), \dots, w_{\rm d}(T)\big) \in (\mathbb{R}^q)^T.$$

The problem studied in [13] is to find for a given finite trajectory $w_{\rm d}$ (the subscript "d" stands for "data") of a linear time-invariant system $\mathscr{B}$ and a given natural number L, $1 \leq L \leq T$, conditions under which the 'windows' of length L

$$\begin{bmatrix} w_{\rm d}(1) \\ \vdots \\ w_{\rm d}(L) \end{bmatrix}, \begin{bmatrix} w_{\rm d}(2) \\ \vdots \\ w_{\rm d}(L+1) \end{bmatrix}, \dots, \begin{bmatrix} w_{\rm d}(T-L+1) \\ \vdots \\ w_{\rm d}(T) \end{bmatrix}, \qquad \text{(WIN)}$$

constructed from the trajectory $w_{\rm d}$ span the space $\mathscr{B}|_L$ of all possible windows of length L which the system can produce. A compact way of writing that (WIN) spans $\mathscr{B}|_L$ is

$$\mathscr{B}|_L = \text{image} \underbrace{\begin{bmatrix} w_{\rm d}(1) & w_{\rm d}(2) & \cdots & w_{\rm d}(T-L+1) \\ \vdots & \vdots & & \vdots \\ w_{\rm d}(L) & w_{\rm d}(L+1) & \cdots & w_{\rm d}(T) \end{bmatrix}}_{\mathscr{H}_L(w_{\rm d})}, \qquad \text{(DD-REPR)}$$

where $\mathscr{H}_L(w_{\rm d})$ is the Hankel matrix, constructed from $w_{\rm d}$ with L block-rows. We refer to (DD-REPR) as the *data-driven representation* of the restricted behavior $\mathscr{B}|_L$. The problem addressed in [13] can then be rephrased as:

> Under what conditions does the data-driven representation (DD-REPR) hold true?

The complexity of a linear time-invariant system $\mathscr{B}$ is characterized by three integers that are properties of the system: the number of inputs $\mathbf{m}(\mathscr{B})$, the lag $\mathbf{l}(\mathscr{B})$, and the order of $\mathbf{n}(\mathscr{B})$ [4, Section III]. In a minimal state-space representation of the system, $\mathbf{n}(\mathscr{B})$ is the state dimension and $\mathbf{l}(\mathscr{B})$ is the observability index. As shown in [4], for $L \geq \mathbf{l}(\mathscr{B})$, a necessary and sufficient condition for (DD-REPR) is

$$\operatorname{rank} \mathscr{H}_L(w_{\rm d}) = \mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B}). \qquad \text{(GPE)}$$

We refer to (GPE) as a *generalized persistency of excitation* condition. This condition can be verified from the data $w_{\rm d}$ and the prior knowledge of the number of inputs and the order of the data-generating system.

The solution given in [13] assumes a given input/output partitioning $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ of the variables and provides sufficient conditions only: (DD-REPR) holds true assuming that

A1: $\mathscr{B}$ is controllable and
A2: $\text{PE}(u_{\rm d}) = L + \mathbf{n}(\mathscr{B})$,

where $\text{PE}(u_{\rm d})$ is the order of persistency of excitation of $u_{\rm d} \in (\mathbb{R}^m)^T$, *i.e.*, the maximal L, for which $\mathscr{H}_L(u_{\rm d})$ is full row-rank. Assumption A1 is not verifiable from the data and Assumption A2 requires prior knowledge of the order of the data-generating system. The need to assume input/output partitioning and controllability as well as the sufficiency but not necessity of A1 and A2 make the result of [13] more restrictive than (GPE). Obtaining conditions for (DD-REPR) in terms of the input $u_{\rm d}$ is not motivated in [13]. A possible motivation for this choice is *input design:* A2 can be used for choosing the input so that the data $w_{\rm d}$ is guaranteed to ensure (DD-REPR) for any initial condition.

In the behavioral setting, initial conditions of a trajectory are specified by a prefix trajectory $w_{\rm d,ini} \in (\mathbb{R}^q)^{T_{\rm ini}}$. Let $w_{\rm d,ini} \wedge w_{\rm d}$ be the concatenation of the trajectories $w_{\rm d,ini}$ and $w_{\rm d}$, ie,

$$w_{\rm d,ini} \wedge w_{\rm d} := \big(w_{\rm d,ini}(1), \dots, w_{\rm d,ini}(T_{\rm ini}), \\ w_{\rm d}(1), \dots, w_{\rm d}(T)\big) \in (\mathbb{R}^q)^{T_{\rm ini}+T}.$$

In $w_{\rm d,ini} \wedge w_{\rm d}$, the prefix $w_{\rm d,ini}$ specifies the initial condition for $w_{\rm d}$ provided $T_{\rm ini} \geq \mathbf{l}(\mathscr{B})$ [6, Lemma 1].

We refine the problem statement of [13] as follows: for given $L \geq \mathbf{l}(\mathscr{B})$, find nonconservative conditions on the input $u_{\rm d}$ and the data-generating system $\mathscr{B}$, under which

$$\text{for any initial condition } w_{\rm d,ini} \in \mathscr{B}|_{T_{\rm ini}}, \text{ a trajectory } \\ w_{\rm d,ini} \wedge w_{\rm d} \in \mathscr{B}|_{T_{\rm ini}} \text{ satisfies (GPE).} \qquad \text{(GOAL)}$$

We verify (GOAL) by proving that there is no initial condition for which (GPE) is not satisfied. Thus, our proof searches for the $w_{\rm ini}$ that minimizes the rank of $\mathscr{H}_L(w_{\rm d})$.

Assumption A1 and the weaker version of Assumption A2:

A2′: $\text{PE}(u_{\rm d}) = L$

are necessary for (GOAL). [1] Moreover, under conditions A1 and A2′, (GOAL) holds generically, *i.e.*, for any controllable system $\mathscr{B}$ and any input $u_{\rm d}$ satisfying A2′, (GPE) holds true for almost all $w_{\rm d,ini}$. Alternatively, (GOAL) holds almost

[1] To see that Assumption A1 is necessary, assume to the contrary that $\mathscr{B}$ is uncontrollable. Then, $\mathscr{B} = \mathscr{B}_{\rm ctr} \oplus \mathscr{B}_{\rm aut}$, where $\mathscr{B}_{\rm ctr}$ is controllable and $\mathscr{B}_{\rm aut}$ is autonomous [11, Proposition V.8]. Therefore, $w_{\rm d} \in \mathscr{B}$ can be decomposed into $w_{\rm d} = w_{\rm d,ctr} + w_{\rm d,aut}$, with $w_{\rm d,ctr} \in \mathscr{B}_{\rm ctr}$ and $w_{\rm d,aut} \in \mathscr{B}_{\rm aut}$. Since $w_{\rm d,aut}$ is completely determined by the initial condition, there is $w_{\rm d,ini}$, such that $w_{\rm d,aut} = 0$ and therefore (GPE) does not hold. To see that Assumption A2′ is necessary, denote with Π_u the projection of $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ on the u-component. Since u is an input, $\Pi_u \mathscr{B}|_L = \mathbb{R}^{\mathbf{m}(\mathscr{B})L}$. Therefore, for (GPE) to hold true, image $\mathscr{H}_L(u_{\rm d}) = \mathbb{R}^{\mathbf{m}(\mathscr{B})L}$ or, equivalently, $\mathscr{H}_L(u_{\rm d})$ must be full row-rank.

certainly for a random controllable system $\mathscr{B}$ and a random input u_d satisfying A2′. The question occurs:

What are the cases of A1 and A2′ in which (GOAL) fails?

The extra persistency of excitation of order $\mathbf{n}(\mathscr{B})$ in Assumption A2 that is needed beyond the obvious persistency of excitation of order L is the crux of the result in [13].

> "The interesting, and somewhat surprising, part of Theorem 1 (the fundamental lemma) is that persistency of excitation of order $L+\mathbf{n}(\mathscr{B})$ is needed in order to be able to deduce that the observed sequences (WIN) of length L have the "correct" annihilators and the "correct" span. In other words, we have to assume a "deeper" persistency of excitation on u_d than the width of the windows of $(u_\mathrm{d}, y_\mathrm{d})$ which are considered." [13, Section 4]

The original publication [13] as well as subsequent publications using and generalizing the result (see, *e.g.*, [8,3,9,10]) do not given explanation (or speculation) that shed light on this crucial fact. In addition, presently it is not known how conservative assumptions A1 and A2 are for (GOAL). We address these questions in the following section.

Summary of notation:

- $w|_L := \big(w(1),\dots,w(L)\big)$ — restriction of w to $[1,L]$
- $\mathscr{B}|_L := \{\, w|_L \mid w \in \mathscr{B} \,\}$ — restriction of $\mathscr{B}$ to $[1,L]$
- $\mathbf{m}(\mathscr{B})$ / $\mathbf{l}(\mathscr{B})$ / $\mathbf{n}(\mathscr{B})$ — number of inputs/lag/order of $\mathscr{B}$
- $\mathscr{H}_{L,j}(w)$ — Hankel matrix with L block rows and j columns (by default $j = T-L+1$, *i.e.*, all data is used)
- $\mathrm{PE}(u)$ — order of persistency of excitation of u
- $\mathscr{B}_\mathrm{ss}(A,B,C,D)$ / $\mathscr{B}_\mathrm{ss}(A,C)$ — minimal state-space representation of input-output / autonomous system
- $\mathscr{C}_L(A,B)$ / $\mathscr{O}_L(A,C)$ — extended controllability/observability matrix with L block-columns/rows
- $\exp_\lambda$ — exponential function, $\exp_\lambda(t) := e^{\lambda t}$

3 Main results

As in [13], we consider a controllable linear time-invariant system $\mathscr{B}$ with an input/output partitioning $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. In Section 3.2 we present an alternative constructive proof of the result of [13] for the single-input case $\mathbf{m}(\mathscr{B}) = 1$. The proof is constructive because the persistency of excitation assumption is derived rather than postulated. The new proof shows that in the single-input case assumptions A1 and A2 are necessary and sufficient for (GOAL). The proof is based on the fact, proven in Section 3.1, that persistency of excitation of $u_\mathrm{d} \in \mathbb{R}^T$ is equivalent to existence of an autonomous linear time-invariant model $\mathscr{B}_u$ of order $n_u = \mathrm{PE}(u_\mathrm{d})$. Then, a state-space representation of the system combining the input model $\mathscr{B}_u$ and the system $\mathscr{B}$ allows us to characterize the nongeneric cases of (GOAL), corresponding to A1 and A2′.

3.1 Model of input with bounded persistency of excitation

Lemma 1 *The following are equivalent:*

(1) $u_\mathrm{d} \in (\mathbb{R})^T$ *is persistently exciting of order* $PE(u_\mathrm{d}) = n_u$,
(2) u_d *is a response of an autonomous system* $\mathscr{B}_u = \mathscr{B}_\mathrm{ss}(A_u, C_u)$ *of order* n_u *with* $T \geq 2n_u - 1$ *samples,* i.e., $u_\mathrm{d} \in \mathscr{B}_u|_T$, *and initial condition* $x_{u,\mathrm{ini}} = x_u(1)$, *such that* $\big(A_u, x_{u,\mathrm{ini}}\big)$ *is controllable.*

PROOF. The condition $u_\mathrm{d} \in \mathscr{B}_\mathrm{ss}(A_u, C_u)$ implies that,

$$\mathscr{H}_L(u_\mathrm{d}) = \mathscr{O}_L(A_u, C_u)\mathscr{C}_{T-L+1}\big(A_u, x_{u,\mathrm{ini}}\big), \text{ for } L \leq T \quad \text{(FAC)}$$

and, vice versa, the factorization (FAC) implies that $u_\mathrm{d} \in \mathscr{B}_\mathrm{ss}(A_u, C_u)|_T$. The matrix $\mathscr{O}_L(A_u, C_u)$ is full row rank for all $L = 1,\dots,n_u$ and $\mathscr{C}_{T-L+1}\big(A_u, x_{u,\mathrm{ini}}\big)$ is full row rank for all $L = 1,\dots,n_u$ due to the minimality of the representation, the controllability of the pair $\big(A_u, x_{u,\mathrm{ini}}\big)$, and the assumption $T \geq 2n_u - 1$. It follows that $\mathscr{H}_L(u_\mathrm{d})$ is full row rank for all $L = 1,\dots,n_u$. Finally, rank $\mathscr{H}_{n_u+1}(u_\mathrm{d}) = n_u$, so that $\mathscr{H}_{n_u+1}(u_\mathrm{d})$ is rank deficient. □

Using Lemma 1, a system $\mathscr{B} = \mathscr{B}_\mathrm{ss}(A,B,C,D)$, whose input u is persistently exciting of order n_u, can be augmented with the "internal model" $\mathscr{B}_u = \mathscr{B}_\mathrm{ss}(A_u, C_u)$ of the input, resulting in an extended autonomous system $\mathscr{B}_\mathrm{ext}$ for $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. The augmented system $\mathscr{B}_\mathrm{ext}$, including the input model $\mathscr{B}_u$ and $\mathscr{B}$ is given by

$$\mathscr{B}_\mathrm{ext} = \mathscr{B}_\mathrm{ss}(A_\mathrm{ext}, C_\mathrm{ext}), \quad \text{where}$$
$$A_\mathrm{ext} = \begin{bmatrix} A_u & 0 \\ BC_u & A \end{bmatrix} \text{ and } C_\mathrm{ext} = \begin{bmatrix} C_u & 0 \\ DC_u & C \end{bmatrix}.$$

The extended state is $x_\mathrm{ext} = \left[\begin{smallmatrix} x_u \\ x \end{smallmatrix}\right]$, where x_u is the state of $\mathscr{B}_\mathrm{ss}(A_u, C_u)$ and x is the state of $\mathscr{B}_\mathrm{ss}(A,B,C,D)$. As a result of using the input model $\mathscr{B}_u$, we transform the original problem about the input/output system $\mathscr{B}$ into an equivalent problem about the autonomous system $\mathscr{B}_\mathrm{ext}$. This is the main motivation of using the input model in the first place.

The following proposition derives a state transformation that block-diagonalizes A_ext. The block-diagonalization leads to a representation of $\mathscr{B}_\mathrm{ext}$ with decoupled states of the input model $\mathscr{B}_u$ and the system $\mathscr{B}$, which simplifies the analysis.

Proposition 1 *Assume that* A *and* A_u *have no common eigenvalues and let* $V \in \mathbb{R}^{n \times n_u}$ *be the solution to the Sylvester equation*

$$AV - VA_u = BC_u. \quad \text{(SYLV)}$$

Then, $\mathscr{B}_{\text{ext}} = \mathscr{B}_{\text{ss}}(A'_{\text{ext}}, C'_{\text{ext}})$, where

$$A'_{\text{ext}} = \begin{bmatrix} A_u & 0 \\ 0 & A \end{bmatrix} \text{ and } C'_{\text{ext}} = \begin{bmatrix} C_u & 0 \\ C' & C \end{bmatrix}, \text{ with } C' := DC_u - CV.$$

The state of $\mathscr{B}_{\text{ss}}(A'_{\text{ext}}, C'_{\text{ext}})$ is $x'_{\text{ext}} = \left[\begin{smallmatrix} x_u \\ Vx_u + x \end{smallmatrix}\right]$, where x_u is the state of $\mathscr{B}_{\text{ss}}(A_u, C_u)$ and x is the state of $\mathscr{B}_{\text{ss}}(A,B,C,D)$.

PROOF. Consider a similarity transformation $\left[\begin{smallmatrix} I_{n_u} & 0 \\ V & I_n \end{smallmatrix}\right]$, where $V \in \mathbb{R}^{n \times n_u}$. We have,

$$\begin{bmatrix} I_{n_u} & 0 \\ V & I_n \end{bmatrix} \begin{bmatrix} A_u & 0 \\ BC_u & A \end{bmatrix} = \begin{bmatrix} A_u & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} I_{n_u} & 0 \\ V & I_n \end{bmatrix},$$

The lower-left block of the equation gives the Sylvester equation (SYLV) for V. The existence and uniqueness of the solution of (SYLV) follows from [2, Theorem 8.2.1.]. □

Proposition 1 shows that the nongeneric cases of A1 and A2′ in which (GOAL) fails correspond to a special choice of the initial condition of $\mathscr{B}$:

$$x_{\text{ini}} = -Vx_{u,\text{ini}}. \tag{NONGEN}$$

Indeed, then $w_{\text{d}} = \left[\begin{smallmatrix} C_u \\ C' \end{smallmatrix}\right] \exp_{A_u} x_{u,\text{ini}}$, so $\operatorname{rank} \mathscr{H}_L(w_{\text{d}}) \leq n_u$. In a trajectory w_{d} corresponding to (NONGEN) the transient is removed, *i.e.*, y_{d}, has no terms $\exp_{\lambda_i}$, where λ_i is an eigenvalue of A. This fact leads us to a constructive proof.

3.2 *Constructive proof*

In view of the necessary condition A2′, let $\text{PE}(u_{\text{d}}) = L + k$, where k will be determined as the minimum natural number, for which (GOAL) holds. By Proposition 1, (GOAL) holds if and only if (GPE) holds for the initial condition (NONGEN). Therefore, we only need to guarantee the rank condition (GPE) for (NONGEN). In what follows, we adopt a proof strategy based on a sum-of-polynomials-times-exponentials representation of w_{d}. In order to simplify the derivation, with some loss of generality, we assume that $\mathscr{B}$ and $\mathscr{B}_u$ have distinct eigenvalues. Then, the proof is effectively using the modal forms of $\mathscr{B}(A,B,C,D)$ and $\mathscr{B}(A_u, C_u)$.

Lemma 2 *Assume that the eigenvalues $\lambda_{u,1}, \ldots, \lambda_{u,n_u}$ of $\mathscr{B}_u$ and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathscr{B}$ are simple and distinct,* i.e., *$\lambda_{u,i} \neq \lambda_j$, for all $i = 1, \ldots, n_u$ and $j = 1, \ldots, n$. Let $u_{\text{d}} \in \mathscr{B}_u|_T$ with $\text{PE}(u_{\text{d}}) = n_u = L + k$ and y_{d} be the corresponding output of $\mathscr{B}$ under the initial conditions (NONGEN). Then,*

$$\operatorname{rank} \mathscr{H}_L(w_{\text{d}}) = \begin{cases} L + k, & \text{for } k = 1, \ldots, n \\ L + n, & \text{for } k = n+1, \ldots \end{cases} \tag{RANK}$$

PROOF. As a corollary of Lemma 1, u_{d} is a sum-of-polynomials-times-exponentials signal, see [7, Thm. 3.2.5]. Since, the eigenvalues of $\mathscr{B}_u$ are simple,

$$u_{\text{d}} = \sum_{i=1}^{n_u} a_i \exp_{\lambda_{u,i}}. \tag{SE}$$

Since the eigenvalues of $\mathscr{B}_u$ are distinct, the rank of the Hankel matrix $\operatorname{rank} \mathscr{H}_{n_u}(u_{\text{d}})$ is equal to the number of nonzero coefficients a_i. On the other hand, $\operatorname{rank} \mathscr{H}_{n_u}(u_{\text{d}}) = n_u$ because $\text{PE}(u_{\text{d}}) = n_u$. Therefore, $a_i \neq 0$ for all $i = 1, \ldots, n_u$.

The output y_{d} corresponding to the input (SE) and general initial conditions is

$$y_{\text{d}} = \sum_{i=1}^{n_u} b_i \exp_{\lambda_{u,i}} + \sum_{j=1}^{n} c_j \exp_{\lambda_j},$$

where $b_i = H(e^{\mathbf{i}\lambda_{u,i}}) a_i$, for $i = 1, \ldots, n_u$, $\mathbf{i} := \sqrt{-1}$, $H(z) := C(Iz - A)^{-1}B + D$, and the c_j's are determined by the initial conditions and the input, see [1, Section 12.12]. With initial conditions (NONGEN), $c_j = 0$ for all $j = 1, \ldots, n$.

Define the Vandermonde matrix

$$V_T(\lambda_u) := \begin{bmatrix} \lambda_{u,1}^1 & \cdots & \lambda_{u,n_u}^1 \\ \vdots & & \vdots \\ \lambda_{u,1}^T & \cdots & \lambda_{u,n_u}^T \end{bmatrix}$$

and let $H(\lambda_u) := \operatorname{diag}\left(H(e^{\mathbf{i}\lambda_{u,1}}), \ldots, H(e^{\mathbf{i}\lambda_{u,n_u}})\right)$. With this notation we rewrite u_{d} and y_{d} as $u_{\text{d}} = V_T(\lambda_u)a$ and $yd = V_T(\lambda_u)H(\lambda_u)a$. For the trajectory $w_{\text{d}} = \left[\begin{smallmatrix} u_{\text{d}} \\ y_{\text{d}} \end{smallmatrix}\right]$, we have

$$w_{\text{d}} = \Pi_T \begin{bmatrix} V_T(\lambda_u) \\ V_T(\lambda_u)H(\lambda_u) \end{bmatrix} a,$$

where $\Pi_T \in \mathbb{R}^{2T \times 2T}$ is the permutation matrix that makes the identity $w_{\text{d}} = \Pi_T \left[\begin{smallmatrix} u_{\text{d}} \\ y_{\text{d}} \end{smallmatrix}\right]$. Then, we have

$$\mathscr{H}_L(w_{\text{d}}) = \underbrace{\Pi_L \begin{bmatrix} V_L(\lambda_u) \\ V_L(\lambda_u)H(\lambda_u) \end{bmatrix}}_{W_L} \underbrace{\begin{bmatrix} a & \Lambda_u a & \Lambda_u^2 a & \cdots & \Lambda_u^{T-L} a \end{bmatrix}}_{\mathscr{C}_{T-L+1}(\Lambda_u, a)},$$

where $\Lambda_u := \operatorname{diag}(\lambda_{u,1}, \ldots, \lambda_{u,n_u})$. The right-hand-side factor matrix $\mathscr{C}_{T-L+1}(\Lambda_u, a)$ is full row rank because $\lambda_{u,i} \neq \lambda_{u,j}$, for all $i \neq j$, and $a_i \neq 0$, for all i. The columns of the left-hand-side factor matrix W_L, viewed as L-sample signals are L-sample long trajectories of $\mathscr{B}$, *i.e.*, they are elements of $\mathscr{B}|_L$. Therefore, image $W_L \subseteq \mathscr{B}|_L$ and, by [5, equation (1)],

$$\dim W_L \leq \dim \mathscr{B}|_L = L + n, \quad \text{for } L \geq \mathbf{n}(\mathscr{B}).$$

The columns of W_L are linearly independence for $k \leq n$ because $\lambda_{u,i} \neq \lambda_{u,j}$, for all $i \neq j$. Then, we have

$$\operatorname{rank} W_L = \begin{cases} L+k, & \text{for } k = 1,\dots,n \\ L+n, & \text{for } k = n+1,\dots \end{cases} \qquad \square$$

Lemma 2 shows that for (GOAL) to hold true for all initial conditions, we need to take $k = n$. This proves that assumptions A1 and A2 are necessary and sufficient for (GOAL).

Theorem 3 *For* $\mathbf{m}(\mathscr{B}) = 1$ *and* $L \geq \mathbf{l}(\mathscr{B})$, *assumptions A1 and A2 are necessary and sufficient for (GOAL).*

Comments

System zeros In the sum-of-exponentials representation of the output y_d up to n coefficients b_i may be equal to zero due to $\lambda_{u,i}$'s matching the zeros of $H(z)$. Therefore, y_d may be a sum of as few as $L+k-n$ exponents. This, however, does not affect (RANK).

Robustifying the persistency of excitation conditions For $\text{PE}(u_d)$ to hold true 1) $a_i \neq 0$, for all i and 2) $\lambda_{u,i} \neq \lambda_{u,j}$, for all $i \neq j$. A way of robustifying these conditions is 1) $a_i > \varepsilon$, for some user defined tolerance ε, and 2) choose the $\lambda_{u,i}$'s "well spread".

Input design using the input model $\mathscr{B}_u$ Using the input model representation of a persistently exciting signal (see, Figure **??**), the freedom of choosing an input that satisfies the conditions of the fundamental lemma is equivalent to choosing the input model $\mathscr{B}_u$ and the initial conditions $x_{u,\text{ini}}$. The input model representation of the class of sufficiently exciting inputs can be used then for input design under user defined specifications, such as frequency band, maximum/minimum value bounds, *etc.*

Simple eigenvalues of $\mathscr{B}_u$ and $\mathscr{B}$ The assumptions of simple and distinct eigenvalues are generic. The proof of Lemma 2 can be generalized to the case of multiple eigenvalues, however, the distinctness of the eigenvalues of A_u and A is still needed for Proposition 1.

Multivariable systems We conjecture that in the multi-input case, the persistency of excitation assumption A2 of the fundamental lemma is conservative and can be improved. Let $\boldsymbol{\ell}_{\text{ctrb}}(\mathscr{B})$ be controllability index of $\mathscr{B}$. We conjecture that assumption A1 and
A2″: $\text{PE}(u_d) = L + \boldsymbol{\ell}_{\text{ctrb}}(\mathscr{B})$
are necessary and sufficient conditions for (GOAL).

4 Simulation example

The purpose of the simulation examples shown in this section is to verify empirically the result of Theorem 3. The simulation parameters are the order `n` of the system $\mathscr{B}$ and the horizon `L` of the data-driven representation (DD-REPR).

```
n = 4; L = n + 1;
```

The number of samples `T` of the trajectory w_d is chosen to be sufficiently large, so that Assumption A2 can hold.

```
T = 3 * (L + n);
```

With these parameters, $\mathscr{B}$ is selected as a random linear time-invariant system.

```
B = drss(n);
```

A random system is almost certainly controllable, so that it satisfies Assumption A1.

In the experiments, we choose the following orders of persistency of excitation `nu` of the input u_d:

E1: `nu = L`, which corresponds to Assumption A2′,
E2: `nu = L + n - 1`, which is the highest order not satisfying Assumption A2,
E3: `nu = L + n`, which corresponds to Assumption A2.

According to Lemma 1, in all experiments the input u_d is generated as an output of an autonomous system $\mathscr{B}_u$ with order `nu`, which is equal to $\text{PE}(u_d)$. The system $\mathscr{B}_u$ is selected as random marginal stable (`drss_ms`). Marginal stability is needed in order to avoid divergence and convergence to 0 of u_d as T grows.

```
Bu = drss_ms(nu, 1, 0);
```

With the systems `B` and `Bu` selected, we are ready to simulate data w_d under different initial conditions and test (GPE). We do the simulation first with random initial condition:

```
x0u = rand(nu, 1); u = initial(Bu, x0u, T-1);
x0  = rand(n, 1);  y = lsim(B, u, [], x0);
```

and then with the special initial condition (NONGEN), which leads to a nongeneric case for (GOAL):

```
v = sylvester(B.a, -Bu.a, B.b * Bu.c);
y_ = lsim(B, u, [], - v * x0u);
```

We verify that $\text{PE}(u) = n_u$

```
Hu = blkhank(u, nu + 1);
check = (rank(Hu) == nu) % -> TRUE
```

For the simulated trajectories `[u,y]` and `[u,y_]`, we check if (GPE) holds.

```
H  = blkhank([u y ], L);
check = (rank(H ) == L + n) % -> TRUE
H_ = blkhank([u y_], L);
check = (rank(H_) == L + n) % -> ???
```

The results for the experiments E1–3 are shown in Table 1. In all experiments $\text{PE}(u) = n_u$. The cases when (GPE) holds confirm empirically Theorem 3, *i.e.*, the fact that Assumption A2 is necessary and sufficient for (GOAL).

Table 1
The results of checking (GPE) with random initial condition (generic case) and the special initial condition (NONGEN) (non-generic case) confirm Theorem 3, *i.e.*, Assumption A2 is necessary and sufficient for (GOAL).

experiment	E1	E2	E2
assumption	A2′	—	A2
PE(u)	L	$L+n-1$	$L+n$
generic x_{ini}	true	true	true
nongeneric x_{ini}	false	false	true

Acknowledgements

I. Markovsky is funded by the Catalan Institution for Research and Advanced Studies (ICREA), FWO projects G090117N and G033822N; and FNRS–FWO EOS Project 30468160. E. Prieto is a lecturer of the Serra Húnter Programme.

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