The most powerful unfalsified linear parameter-varying model

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Abstract

The most powerful unfalsified model (MPUM), *i.e.*, the least complex exact model for the given data, is well established for linear time-invariant (LTI) systems. It has not been generalized for linear parameter-varying (LPV) systems. In order to do this, we define the notions of complexity for LPV systems with shifted-affine scheduling dependence. The MPUM leads to identifiability conditions and a method for exact LPV system identification. The method is based on lifting the LPV system to a higher dimensional space and LTI embedding in the lifted space. It is made rigorous by proving a formal connection between the parameters of the LTI embedding and the original LPV system.

1 Introduction

The *most powerful unfalsified model (MPUM)* of a vector time series, *i.e.*, the least complex exact model in a given class of systems for data consisting of vector time series, laid a clear and rational foundation for system identification [20]. The problem of finding the MPUM is a generalization of the partial realization problem, when the data is a general trajectory of the system rather than the impulse response. The MPUM was defined originally for the class of *linear time-invariant (LTI)* systems and data being an infinite time series. Later on, it was used to generalize and give a system theoretic interpretation of the Berlekamp-Massey algorithm and was modified for the case of finite time series [6, Section IV] and time series with missing values [4,1]. It has not been defined and used for other classes of systems beyond LTI, the main issue being the generalization of the notion of model complexity.

The problem of finding the MPUM is an exact (deterministic) identification problem. The algorithms proposed in [20] inspired the development of the N4SID subspace identification methods [14]. In particular, the fact that a state sequence of a state-space realization of an unknown system can be obtained from a trajectory of the system was proposed in [20] as a method for computing the MPUM.

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In this communique, we generalize the notion of the MPUM for the class of *linear parameter-varying (LPV)* systems, *i.e.*, linear systems whose behavior is defined by a linear relation that depends on a signal, called the *scheduling signal* [11]. The LPV model class is a generalization of the LTI one and can be viewed as an intermediate step towards general non-linear systems [18]. The scheduling variable is often assumed measurable.

In this work, the laws describing the LPV system are assumed to have a representation with an affine dependence on the scheduling variable (see (2) and (5)). We call this subclass *shifted-affine LPV (SALPV)*. It has been shown that the SALPV model class has a minimal state-space realization with affine dependence [12], which makes it directly applicable for control design as it has been also demonstrated in [3]. Furthermore, recent developments in direct data-driven LPV controller synthesis are based on the SALPV model class and are successfully applied in practice [17].

Our objective is to define the MPUM for the class of SALPV systems and find methods for its computation. To this end, we transform the SALPV identification problem to a problem that is similar to LTI system identification, by a lifting transformation: a Kronecker product of the manifest variable and the scheduling variable. The connection between the SALPV and the lifted LTI system is established via the kernel representations of the original and lifted systems.

We use the behavioral approach, *i.e.*, we view dynamical systems as sets of trajectories [21,7]. Although *per se* the behavioral approach is applicable to nonlinear as well as linear systems and to time-varying as well as time-invariant systems, there have been only a few attempts to develop behavioral systems theory for nonlinear and/or time-varying systems, *e.g.*, LPV systems [13,11], external representations of nonlinear systems [8], and convex conical behaviors [9]. Next, we review the behavioral setting for LPV systems.

A parameter-varying system $\mathscr{B}$ is a dynamical system, whose variables are partitioned into a *scheduling variable* ρ and a *manifest variable* w. Let $\Pi_w \left[\begin{smallmatrix}\rho\\ w\end{smallmatrix}\right] := w$ be the projection on the manifest variable and $\Pi_\rho \left[\begin{smallmatrix}\rho\\ w\end{smallmatrix}\right] := \rho$ be the projection on the scheduling variable. The manifest behavior $\Pi_w\mathscr{B}$ for a fixed, but not necessarily constant scheduling signal ρ is denoted by $\mathscr{B}_\rho$. If $\mathscr{B}_\rho$ is linear for all $\rho \in \Pi_\rho\mathscr{B}$, the system $\mathscr{B}$ is called LPV. For a given ρ, the projected behavior $\mathscr{B}_\rho$ is *linear time-varying (LTV)*. We will refer to $\mathscr{B}_\rho$ as the "behavior along ρ". We consider discrete-time LPV systems with *unrestricted scheduling variable*, *i.e.*, $\Pi_\rho\mathscr{B} = (\mathbb{R}^{n_\rho})^{\mathbb{T}}$. The time-axis $\mathbb{T}$ is the one-sided infinite $[1,\infty)$ or a finite $[1,T]$ interval.

The main contributions of this short communique are: i) a novel characterization of the finite-horizon behaviour $\mathscr{B}_\rho|_T$ of an general LPV system $\mathscr{B}$ in terms of a kernel representation (Section 2.1), ii) definition and computational method for the MPUM in the SALPV model class (Section 2.2), and iii) identifiability condition for SALPV systems that is verifiable from the data and the complexity of the data-generating system (Theorem 2). Compared to previous work on realization theory for LPV systems, mostly work on state-space realization exists, such as [11] based on a shift-and-cut operation requiring symbolic solution of a set of algebraic relations, [12] that provides simple realization schemes for special cases, and [10], a realization theory for systems that have state-space representations with affine dependence was worked out using impulse response (Markov) coefficients. In this paper, we estimate a kernel representation of the system from data. Although, not all systems with state-space representation of affine dependence have such a representation, computing the MPUM opens up the door for deriving subspace methods with reduced memory needs which could resolve the current computational bottlenecks of LPV subspace identification approaches.

2 Scheduling dependent kernel representation

First, we present the kernel representation of a general LPV system. Then, we introduce the class of LPV systems with shifted-affine scheduling dependence.

2.1 General case

Consider a *finite-dimensional* LPV system $\mathscr{B}$ with n_ρ scheduling variables and q manifest variables. Then, there is a natural number ℓ and functions

$$R_i : (\mathbb{R}^{n_\rho})^{\mathbb{N}} \times \mathbb{T} \to \mathbb{R}^{g\times q}, \quad \text{for } i = 0,1,\ldots,\ell, \qquad (1)$$

such that for any scheduling signal $\rho \in (\mathbb{R}^{n_\rho})^{\mathbb{N}}$, the manifest behavior $\mathscr{B}_\rho$ admits a representation

$$\underbrace{R_0(\rho,t)w(t) + R_1(\rho,t)w(t+1) + \cdots + R_\ell(\rho,t)w(t+\ell)}_{=:\big(R(\rho,\sigma)w\big)(t)} = 0, \qquad (2)$$

where $(\sigma w)(t) := w(t+1)$, is the unit shift operator. The manifest behavior $\mathscr{B}_\rho$ of $\mathscr{B}$ is therefore given as the kernel of the difference operator $R(\rho,\sigma)$:

$$\mathscr{B}_\rho = \ker R(\rho,\sigma) := \{\, w \mid (2) \text{ holds} \,\}. \qquad (3)$$

Vice versa, for any set of parameters (1), the kernel representation (3) defines an LPV system

$$\mathscr{B} = \{\, (\rho,w) \mid \rho \in (\mathbb{R}^{n_\rho})^{\mathbb{N}} \text{ and } w \in \mathscr{B}_\rho \,\}.$$

Next, we consider a finite horizon $\mathbb{T} := \{1,\ldots,T\}$ and denote by $w|_T$ the restriction of $w \in (\mathbb{R}^q)^{\mathbb{N}}$ to $\mathbb{T}$, *i.e.*,

$$w|_T := \big(w(1),\ldots,w(T)\big).$$

For $t \in \mathbb{T}$, (2) is equivalent to the equation $\mathscr{M}_T(R,\rho)w = 0$, where $\mathscr{M}_T(R,\rho) \in \mathbb{R}^{g(T-\ell)\times qT}$ is the polynomial multiplication, defined in Figure 1. In the LTI case, $\mathscr{M}_T(R,\rho)$ additionally has a block-Toeplitz structure and is called the multiplication matrix as it represents polynomial multiplication [6, Section II.B]. This leads us to the characterization of the finite horizon behavior $\mathscr{B}_\rho|_T := \{\, w|_T \mid w \in \mathscr{B}_\rho \,\}$ as the kernel of $\mathscr{M}_T(R,\rho)$

$$\mathscr{B}_\rho|_T = \ker \mathscr{M}_T(R,\rho). \qquad (4)$$

Formula (4) is a generalization to the LPV case of formula ($\mathscr{B}|_L$, KER) in [6], which gives an explicit matrix characterization of the finite horizon behavior of LTI systems. As in the LTI case, (4) is directly amenable to matrix computations. Indeed, given (1) and $\rho \in (\mathbb{R}^{n_\rho})^T$, $\mathscr{M}_T(R,\rho)$ can be evaluated. Consequently, a basis for $\mathscr{B}_\rho|_T$ can be computed by standard numerical linear algebra methods. The availability of $\mathscr{B}_\rho|_T$, in turn, opens the path of solving analysis, noise filtering, and control problems, using the methods presented in [5].

The matrix characterization (4) of $\mathscr{B}_\rho|_T$ is applicable for general LPV systems, requiring only the kernel parameters (1) of the system and the scheduling signal ρ.

2.2 Shifted-affine scheduling dependence

Next, we consider two special cases: 1) LTI systems and 2) SALPV systems. In the LTI case, the parameters (1) are constant in time. Then, (3) becomes the classical kernel representation defined by a polynomial matrix $R(z) := R_0 +$

Fig. 1. Definition of the multiplication matrix.

$$\mathscr{M}_T(R,\rho) := \begin{bmatrix} R_0(\rho,1) & R_1(\rho,1) & \cdots & R_\ell(\rho,1) & & & \\ & R_0(\rho,2) & R_1(\rho,2) & \cdots & R_\ell(\rho,2) & & \\ & & \ddots & \ddots & & \ddots & \\ & & & R_0(\rho,T-\ell) & R_1(\rho,T-\ell) & \cdots & R_\ell(\rho,T-\ell) \end{bmatrix} \in \mathbb{R}^{g(T-\ell)\times qT}.$$

$R_1 z + \cdots + R_\ell z^\ell$ and $\mathscr{M}_T(R,\rho)$ is the polynomial multiplication matrix for $R(z)$. In the SALPV case, considered later in the communique, the parameters (1) are

$$R_i(\rho,t) = R_i^0 + R_i^1 \rho_1(t+i) + \cdots + R_i^{n_\rho} \rho_{n_\rho}(t+i), \quad (5)$$

for $i = 0,1,\ldots,\ell$, where the dependence of $w(t+i)$ on the scheduling signal is limited to $\rho(t+i)$, *i.e.*, the time dependence of the scheduling signal shifts along with the manifest variable. The class of SALPV systems with n_ρ scheduling variables and q manifest variables is denoted by $\mathscr{P}^{n_\rho,q}$. When the dimensions n_ρ and q are understood from the context, they will be skipped from the notation $\mathscr{P}^{n_\rho,q}$.

For an SALPV system $\mathscr{B} \in \mathscr{P}^{n_\rho,q}$, with $\rho_{\text{ext}} := \left[\begin{smallmatrix}1\\ \rho\end{smallmatrix}\right]$,

$$R_i\big(\rho(t+i)\big) = \underbrace{\begin{bmatrix} R_i^0 & R_i^1 & \cdots & R_i^{n_\rho}\end{bmatrix}}_{R_i'} \big(\rho_{\text{ext}}(t+i) \otimes I_q\big), \quad (6)$$

where $\otimes$ is the Kronecker product. We define the matrix $R' := \begin{bmatrix} R_0' & R_1' & \cdots & R_\ell' \end{bmatrix}$, which completely characterizes (3) in the case of shifted-affine scheduling dependence.

3 Exact identification of SALPV systems

3.1 SALPV model complexity

In the behavioral setting, the complexity of a system $\mathscr{B}$ characterizes the "size" of the set $\mathscr{B}$, *i.e.*, the larger the set, the more complex the system is. This intuitive idea is formalized for the case of LTI systems by the triple [6]

$$\mathbf{c}(\mathscr{B}) := \big(\mathbf{m}(\mathscr{B}), \mathbf{l}(\mathscr{B}), \mathbf{n}(\mathscr{B})\big), \quad (7)$$

where $\mathbf{m}(\mathscr{B})$ is the number of inputs of $\mathscr{B}$, $\mathbf{l}(\mathscr{B})$ is the lag of $\mathscr{B}$, and $\mathbf{n}(\mathscr{B})$ is the order of $\mathscr{B}$. [1] The rationale for defining the complexity of $\mathscr{B}$ via (7) is the formula

$$\dim \mathscr{B}|_T = \mathbf{m}(\mathscr{B})T + \mathbf{n}(\mathscr{B}), \quad \text{for } T \geq \mathbf{l}(\mathscr{B}),$$

showing that the dimension of the finite horizon behavior for "large enough" horizon is completely determined by $\mathbf{c}(\mathscr{B})$.

[1] In the LTI case, the number of inputs $\mathbf{m}(\mathscr{B})$, lag $\mathbf{l}(\mathscr{B})$, and order $\mathbf{n}(\mathscr{B})$ are properties of the system $\mathscr{B}$ and are invariant of its representations [19].

As in the LTI case, for an LPV system $\mathscr{B}$, the functions $\mathbf{m}(\mathscr{B})$, $\mathbf{l}(\mathscr{B})$ and $\mathbf{n}(\mathscr{B})$ are also well defined [11]. For a general LPV system, the complexity is determined by the triple $\big(\mathbf{m}(\mathscr{B}), \mathbf{l}(\mathscr{B}), \mathbf{n}(\mathscr{B})\big)$ as well as the complexity of the functions in (1). The simplest nontrivial case of (1) is affine, *i.e.*, the SALPV class. Affine functions (1) have complexity that depends on the dimension n_ρ of the scheduling variable only. Thus, the complexity of $\mathscr{B} \in \mathscr{P}^{n_\rho,q}$ is solely determined by the triple $\big(\mathbf{m}(\mathscr{B}), \mathbf{l}(\mathscr{B}), \mathbf{n}(\mathscr{B})\big)$. We define the complexity of an SALPV system by (7) and the subclass of SALPV systems $\mathscr{P}^{n_\rho,q}$ with complexity bounded by (m,ℓ,n) as

$$\mathscr{P}^{n_\rho,q}_{m,\ell,n} := \{\, \mathscr{B} \in \mathscr{P}^{n_\rho,q} \mid \mathbf{c}(\mathscr{B}) \leq (m,\ell,n) \,\}.$$

3.2 MPUM in the class of SALPV systems

The MPUM for given data in a specified model class is the least complex system in the model class that fits the data exactly. The MPUM depends on the model class as well as the notion of complexity. For given data $(\rho_{\text{d}}, w_{\text{d}})$, we denote by $\text{mpum}(w_{\text{d}})$ the MPUM (of w_{d}) in the LTI model class $\mathscr{L}^q$ and by $\text{mpum}(\rho_{\text{d}}, w_{\text{d}})$ the MPUM (of $(\rho_{\text{d}}, w_{\text{d}})$) in the SALPV model class $\mathscr{P}^{n_\rho,q}$.

For infinite data $w_{\text{d}} \in (\mathbb{R}^q)^{\mathbb{T}}$,

$$\text{mpum}(w_{\text{d}}) = \text{span}\{\, \sigma^t w_{\text{d}} \mid t \in \mathbb{N} \,\}, \quad (8)$$

i.e., $\text{mpum}(w_{\text{d}})$ is the span of w_{d} and all its shifts. The construction (8) can be viewed as an *LTI embedding* of the data w_{d}. More generally, given a set of infinite trajectories $\mathscr{B} \subset (\mathbb{R}^q)^{\mathbb{T}}$, its LTI embedding is defined as

$$\text{LTI}(\mathscr{B}) := \text{span}\{\, \sigma^t w \mid t \in \mathbb{N} \text{ and } w \in \mathscr{B} \,\}.$$

$\text{LTI}(\mathscr{B})$ is the smallest LTI system that includes $\mathscr{B}$.

LTI embedding (8) is a procedure for construction of the MPUM in the class of LTI systems. It can not be used in the SALPV case. One difficulty is that $\mathscr{B}_{\rho_{\text{d}}}$, for which there is data w_{d}, is not time-invariant. Another one is that there is no given data for other scheduling sequences. For a general LPV system, these difficulties make the problem of computing the MPUM ill-posed. For SALPV systems, however, $\text{mpum}(\rho_{\text{d}}, w_{\text{d}})$ is well-defined and therefore it can be constructed. Next, we present an algorithm for doing this.

3.3 The lifting operation

The key idea of finding $\mathrm{mpum}(\rho_d, w_d)$ is lifting the data. For signals $\rho \in (\mathbb{R}^{n_\rho})^{\mathbb{T}}$ and $w \in (\mathbb{R}^{q})^{\mathbb{T}}$, define the Kronecker product $\rho \otimes w$, point-wise in time:

$$(\rho \otimes w)(t) := \rho(t) \otimes w(t) \in \mathbb{R}^{n_\rho q}, \quad \text{for } t \in \mathbb{T}$$

and the "lifting" of $(\rho, w) \in (\mathbb{R}^{n_\rho + q})^{\mathbb{T}}$ as

$$\mathrm{lift}_P(\rho, w) := \left[\begin{smallmatrix} 1 \\ \rho \end{smallmatrix}\right] \otimes w \in (\mathbb{R}^{(1+n_\rho)q})^{\mathbb{T}}. \tag{9}$$

The word "lifting" in the name of $\mathrm{lift}_P(\rho, w)$ refers to the fact that it increases the dimension: the lifted signal $\mathrm{lift}_P(\rho, w)$ has dimension $q' := (1 + n_\rho)q$.

Applied on an LPV system $\mathscr{B}$, the "lift_P" operation acts on all signals in $\mathscr{B}$, producing another system:

$$\mathrm{lift}_P(\mathscr{B}) := \left\{ \mathrm{lift}_P(\rho, w) \mid \left[\begin{smallmatrix} \rho \\ w \end{smallmatrix}\right] \in \mathscr{B} \right\}.$$

The lifting and LTI embedding are used in the next section for the construction of $\mathrm{mpum}(\rho_d, w_d)$, see Fig. 2 for the relationship between $\mathscr{B}$, $\mathrm{lift}_P(\mathscr{B})$, and the LTI embedding of $\mathrm{lift}_P(\mathscr{B})$. Similar signal lifting, without formal characterization of the lifted behavior, has been considered extensively in the LPV subspace literature, see for example [12,16,15].

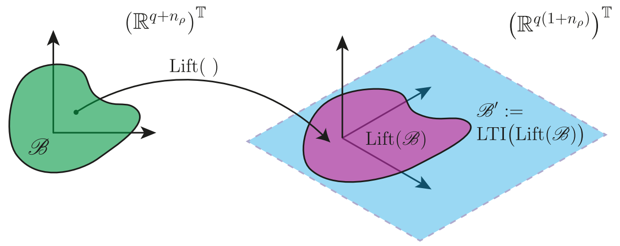

Fig. 2. The LTI embedding of $\mathscr{B} \in \mathscr{P}^{n_\rho, q} \subset (\mathbb{R}^{n_\rho + q})^{\mathbb{T}}$ is applied on the lifted system $\mathrm{lift}_P(\mathscr{B}) \subset (\mathbb{R}^{(n_\rho+1)q})^{\mathbb{T}}$.

3.4 $\mathrm{mpum}(\rho_d, w_d)$ computational method

The method for computing $\mathrm{mpum}(\rho_d, w_d)$ is: 1) construct the lifted signal $w' := \mathrm{lift}_P(\rho, w)$, 2) find the LTI embedding $\mathscr{B}' := \mathrm{mpum}(w')$, and 3) recover $\mathscr{B} := \mathrm{mpum}(\rho_d, w_d)$ from $\mathscr{B}'$. Step 3 relies on the bijective relation

$$\begin{aligned} R'(z) = R'_0 + R'_1 z + \cdots + R'_\ell z^\ell, \quad &\text{with} \\ R'_i = \begin{bmatrix} R_i^0 & R_i^1 & \cdots & R_i^{n_\rho} \end{bmatrix}, \; &\text{for } i = 0, 1, \ldots, \ell \end{aligned} \tag{10}$$

between a kernel representation $\mathscr{B}_\rho = \ker R(\rho, \sigma)$ of an SALPV system $\mathscr{B}$ and a kernel representation $\ker R'(\sigma)$ of the LTI embedding of the lifted system $\mathrm{lift}_P(\mathscr{B})$.

Lemma 1 *Consider an SALPV system $\mathscr{B} \in \mathscr{P}^{n_\rho, q}_{(m,\ell,n)}$ with a kernel representation $\mathscr{B}_\rho = \ker R(\rho, \sigma)$ and define R' via (10). Then, $\mathrm{LTI}\big(\mathrm{lift}_P(\mathscr{B})\big) = \ker R'(\sigma)$.*

PROOF. Substituting (6) in (2) and defining $w'_d := \mathrm{lift}_P(\rho_d, w_d)$, we obtain the linear constant-coefficients-based difference equation

$$R'_0 w'_d(t) + R'_1 w'_d(t+1) + \cdots + R'_\ell w'_d(t+\ell) = 0, \tag{11}$$

which defines an LTI system $\mathscr{B}' := \ker R'(\sigma)$. By construction, for any $(\rho_d, w_d) \in \mathscr{B}$, the lifted signal $w'_d \in \mathscr{B}'$, so that $\mathrm{lift}_P(\mathscr{B}) \subseteq \ker R'(\sigma)$. Moreover, $\mathscr{B}'$ is the smallest LTI system containing $\mathrm{lift}_P(\mathscr{B})$, so that $\mathscr{B}' = \mathrm{LTI}\big(\mathrm{lift}_P(\mathscr{B})\big)$. □

Step 3 of the method for computing $\mathrm{mpum}(\rho_d, w_d)$, is thus the conversion (10) of the kernel parameter R' defining $\mathscr{B}'$ to the parameter (1) defining a kernel representation (3) of the MPUM. Although, $\mathrm{lift}_P(\mathscr{B})$ and $\mathscr{B}'$ are *not* equal, they share the same kernel parameter R', so that $\mathscr{B}$ can be inferred from $\mathscr{B}'$, which in turn can be inferred from the lifted data by the LTI embedding procedure.

Note 1 (Discrepancy between $\mathrm{lift}_P(\mathscr{B})$ and $\mathscr{B}'$) *Not every trajectory $w' \in \mathscr{B}'$ is a trajectory of $\mathrm{lift}_P(\mathscr{B})$,* cf., *Fig. 2. The discrepancy between $\mathrm{lift}_P(\mathscr{B})$ and $\mathscr{B}'$ is due to the structure of (9). More specifically, $\mathrm{lift}_P(\mathscr{B}) = \mathscr{B}' \cap \mathscr{B}_{str}$, where $\mathscr{B}_{str} := \{ \mathrm{lift}_P(\rho, w) \mid \rho \in (\mathbb{R}^{n_\rho})^{\mathbb{T}} \text{ and } w \in (\mathbb{R}^q)^{\mathbb{T}} \}$.*

Note 2 (Complexity of the LTI embedding) *The complexity of the LTI embedding $\mathscr{B}'$ is*

$$\mathbf{c}(\mathscr{B}') = \big(\underbrace{\mathbf{m}(\mathscr{B}) + n_\rho q}_{m'}, \mathbf{l}(\mathscr{B}), \mathbf{n}(\mathscr{B})\big). \tag{12}$$

Intuitively, the LTI embedding relaxes the $n_\rho q$ variables $\rho \otimes w$ of $\mathrm{lift}_P(\mathscr{B})$ by treating them as inputs (free variables). This is equivalent to removing the constraint $w' \in \mathscr{B}_{str}$.

For a known dimension n_ρ of the scheduling variable ρ_d, there is a one-to-one map between the complexity of an SALPV system and the complexity of its LTI embedding.

Note 3 *In case of finite data, define the Hankel matrix*

$$\mathscr{H}_L(w_d) := \begin{bmatrix} w_d|_L & (\sigma w_d)|_L & \cdots & (\sigma^{T_d - L} w_d)|_L \end{bmatrix}.$$

Then, assuming that the lag ℓ of the MPUM is a priori given, a kernel representation of $\mathscr{B}'$ can be computed from the left null space of the Hankel matrix *$\mathscr{H}_{\ell+1}(w'_d)$. When the lag ℓ is unknown, it can be found from w_d by a rank test as in the LTI case [6, Section IV].*

The relation (10) between the parameters of an SALPV system $\mathscr{B}$ and the LTI embedding of $\mathrm{lift}_P(\mathscr{B})$ is the reason why subspace approaches using explicitly or implicitly lifting, such as the Ho-Kalman algorithm of [12], early LPV subspace methods in [16], and recent state-of-the-art subspace approaches [15,2] work. The developed theory in this paper (combined with [10]) shows that such methods implicitly assume the shifted affine dependence on the scheduling variable and via lifting they embed the behavior of the LPV system in an LTI system, for which application of LTI realization theory leads to a representation. By using the result of that representation, it is possible to perform a reformulation to obtain a representation of the LPV system with static affine dependence on the scheduling variable. Hence, our results show the core reasons why the extension of LTI subspace algorithm is possible to LPV systems and generalizes the implicit concepts of these papers towards a general theory of LPV behaviors and subspace identification methods.

3.5 *Identifiability condition*

For any $(\rho_\mathrm{d}, w_\mathrm{d}) \in (\mathbb{R}^{n_\rho})^{T_\mathrm{d}} \times (\mathbb{R}^q)^{T_\mathrm{d}}$, the system $\mathrm{mpum}(\rho_\mathrm{d}, w_\mathrm{d})$ exists and is unique. In the *identifiability problem,* considered in this section, it is assumed that the data $(\rho_\mathrm{d}, w_\mathrm{d})$ is generated by a bounded complexity LPV system $\mathscr{B} \in \mathscr{P}^{n_\rho, q}_{m,\ell,n}$, *i.e.*, $(\rho_\mathrm{d}, w_\mathrm{d}) \in \mathscr{B}|_{T_\mathrm{d}}$. The problem is to find $\mathscr{B}$ back from the data $(\rho_\mathrm{d}, w_\mathrm{d})$. The following result gives conditions under which this is possible. Moreover, under the specified condition, the data generating system coincides with the MPUM, so that using the algorithm for the computation of $\mathrm{mpum}(\rho_\mathrm{d}, w_\mathrm{d})$, presented in Section 3.4, the result becomes constructive.

Theorem 2 *Consider an SALPV system $\mathscr{B} \in \mathscr{P}^{n_\rho, q}_{m,\ell,n}$ and let $(\rho_\mathrm{d}, w_\mathrm{d}) \in \mathscr{B}|_{T_\mathrm{d}}$ be a trajectory of $\mathscr{B}$. Under the generalized persistency of excitation condition*

$$\operatorname{rank} \mathscr{H}_{\ell+1}\big(\mathit{lift}_P(\rho_\mathrm{d}, w_\mathrm{d})\big) = \big((n_\rho+1)q - \mathbf{p}(\mathscr{B})\big)\big(\ell+1\big) + \mathbf{n}(\mathscr{B}), \quad (13)$$

$\mathscr{B}$ is identifiable from $(\rho_\mathrm{d}, w_\mathrm{d})$, i.e., $\mathrm{mpum}(\rho_\mathrm{d}, w_\mathrm{d}) = \mathscr{B}$.

PROOF. By [6, Theorem 17], under the generalized persistency of excitation condition (13), the LTI embedding of the lifted data-generated system $\mathscr{B}$ is identifiable, *i.e.*, $\mathrm{mpum}\big(\mathrm{lift}_P(\rho_\mathrm{d}, w_\mathrm{d})\big) = \mathrm{LTI}\big(\mathrm{lift}_P(\mathscr{B})\big)$. Then, by Lemma 1, if $\mathrm{mpum}\big(\mathrm{lift}_P(\rho_\mathrm{d}, w_\mathrm{d})\big) = \ker R'(\sigma)$ and R is defined via (10), we have that $\ker R(\rho, \sigma) = \mathscr{B}_\rho$, for all $\rho \in (\mathbb{R}^{n_\rho})^{\mathbb{T}}$. □

4 Conclusions

We generalized the notion of the MPUM from LTI to LPV systems with shifted-affine scheduling dependence. The MPUM led us to an identifiability condition that is verifiable from the data and the complexity of the true system. The result is constructive and leads to a method for exact identification that yields a kernel representation of the MPUM. Using the results from approximate identification of SALPV systems as well as efficient solution methods exploiting the Kronecker-Hankel structure is a topic of current research.

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