Realization and identification of autonomous linear periodically time-varying systems

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Abstract

Subsampling of a linear periodically time-varying system results in a collection of linear time-invariant systems with common poles. This key fact, known as "lifting", is used in a two step realization method. The first step is the realization of the time-invariant dynamics (the lifted system). Computationally, this step is a rank-revealing factorization of a block-Hankel matrix. The second step derives a state space representation of the periodic time-varying system. It is shown that no extra computations are required in the second step. The computational complexity of the overall method is therefore equal to the complexity for the realization of the lifted system. A modification of the realization method is proposed, which makes the complexity independent of the parameter variation period. Replacing the rank-revealing factorization in the realization algorithm by structured low-rank approximation yields a maximum likelihood identification method. Existing methods for structured low-rank approximation are used to identify efficiently linear periodically time-varying system. These methods can deal with missing data.

Key words: linear periodically time-varying systems, lifting, realization, Kung's algorithm, Hankel low-rank approximation, maximum likelihood estimation.

1 Introduction

1.1 Overview of the literature

Periodically time-varying systems, i.e., systems with periodic coefficients, appear in many applications and have been studied from both theoretical as well as practical perspectives. The source of the time-variation can be rotating parts in mechanical systems Bittanti and Colaneri (2008); hearth beat and/or breathing in biomedical applications Ionescu *et al.* (2010); Sanchez *et al.* (2013); and seasonality in econometrics Ghysels (1996); Osborn (2001). Linear periodically time-varying systems also appear when a nonlinear system is linearized about a periodic trajectory Sracic and Allen (2011).

In this paper, we restrict our attention to the subclass of discrete-time autonomous linear periodically time-varying systems. A specific application of autonomous linear periodically time-varying system identification in mechanical engineering is vibration analysis, also known as operational modal analysis, see, e.g., Allen and Ginsberg (2006); Allen *et al.* (2011). The problems considered in the paper are exact (Section 2, Problem 1) and approximate (Section 5, Problem 2) identification. The exact identification of an autonomous linear periodically time-varying system is equivalent to realization of an input-output linear periodically time-varying system from impulse response measurement. The approximate identification problem yields a maximum-likelihood estimator in the output error model.

Input-output identification methods for linear periodically time-varying systems are proposed in Hench (1995); Verhaegen and Yu (1995); Liu (1997); Mehr and Chen (2002); Yin and Mehr (2010); Xu *et al.* (2012). Less attention is devoted to the autonomous identification problem. A method for exact identification, based on polynomial algebra, is proposed in Kuijper (1999) and a frequency domain method for output-only identification is developed in Allen (2009); Allen and Ginsberg (2006). Both the method of Kuijper (1999); Kuijper and Willems (1997) and the method of Allen (2009) are based on a lifting approach, i.e., the time-varying system is represented equivalently as a multivariable time-invariant system. The number of outputs p′ of the lifted system is equal to the number of outputs p of the original periodic system times the number of samples P in a period of the parameter variation.

1.2 Aim and contribution of the paper

Most methods proposed in the literature consist of the following main steps (see also Figure 1):

(1) *preprocessing* — lifting of the data,
(2) *main computation* — derivation of a linear time-invariant model for the lifted data,
(3) *postprocessing* — derivation of an equivalent linear periodically time-varying model.

The key in solving the linear periodic time-varying realization and identification problem is the lifting operation, which converts the time-varying dynamics into time-invariant dynamics of a system with $\mathrm{p}' = \mathrm{p}P$ outputs. From a computational point of view, the realization of the lifted dynamics is a rank-revealing factorization of a block-Hankel matrix. A numerically stable way of doing this operation is the singular value decomposition of a $\mathrm{p}'L \times (T − L)$ matrix, where L is an upper bound on the order, p is the number of outputs, and T
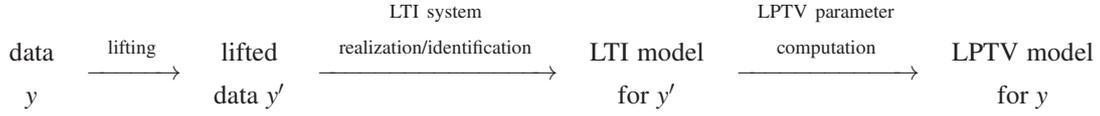
Fig. 1. Main steps of the linear periodic time-varying system realization/identification methods. (LTI — linear time-invariant, LPTV — linear periodically time-varying)

is the number of time samples. Its computational complexity is $O(L^2 \mathrm{p}^2 P T)$ operations.

Once the linear time-invariant dynamics of the lifted model is obtained, it is transformed back to a linear periodically time-varying model in a postprocessing step. In the subspace identification literature, see, e.g., Hench (1995), this operation is done indirectly by computing shifted versions of the state sequence of the model and solving linear systems of equations for the model parameters. This method, refered to as the "indirect method" is Algorithm 1 in the paper, has computational complexity $O(L^2 \mathrm{p}^2 P^2 T)$.

The main shortcoming of the indirect method is that it requires extra computations for the derivation of the shifted state sequences and the solution of the systems of equations for the model parameters. This increases the computational complexity by a factor of P compared with the complexity of the realization of the lifted system. We show in Section 4 that the linear periodically time-varying model's parameters can be obtained directly from the lifted model's parameters without extra computations. The resulting method, refered to as the "direct method" is Algorithm 2 in the paper. Its computational complexity is $O(L^2 \mathrm{p}^2 P T)$. A further improvement of the indirect method (Algorithm 3) operates on a $L \times \mathrm{p}(T - L)$ Hankel matrix and requires $O(L^2 \mathrm{p} T)$ operations.

The maximum-likelihood estimation problem is considered in Section 5. Using the results relating the realization problem to rank revealing factorization of a Hankel matrix constructed from the data, we show that the maximum-likelihood identification problem is equivalent to Hankel structured low-rank approximation. Subsequently, we use existing efficient local optimization algorithms Usevich and Markovsky (2013) for solving the problem.

The motivation for reformulating the maximum likelihood identification problem as structured low-rank approximation is the possibility to use readily available solution methods. Structured-low-rank approximation is an active area of research that offers a variety of solution methods, e.g., convex relaxation methods, based on the nuclear norm heuristic. There are also methods for solving problems with missing data Markovsky and Usevich (2013). Identification with missing data is a challenging problem, however, using the link between system identification and low-rank approximation, identification of autonomous periodically time-varying systems with missing data becomes merely an application of existing methods.

The main contributions of the paper are summarized next.

(1) Reduction of the computational cost of linear periodically time-varying system realization from $O(L^2 \mathrm{p}^2 P^2 T)$ to $O(L^2 \mathrm{p} T)$.

(2) Maximum-likelihood method for linear periodically time-varying system identification with computational complexity per iteration that is linear in the number of data points. In addition, the maximum-likelihood method can deal with missing data.

2 Preliminaries, problem formulation, and notation

An autonomous discrete-time linear time-varying system $\mathscr{B}$ can be represented by a state space model

$$\mathscr{B} = \mathscr{B}(A,C) := \{\, y \mid x(t+1) = A(t)x(t), y(t) = C(t)x(t),$$
$$\text{for all } t, \text{ with } x(1) = x_{\mathrm{ini}} \in \mathbb{R}^{\mathrm{n}} \,\}, \quad (1)$$

where $A(t) \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$ and $C(t) \in \mathbb{R}^{\mathrm{p} \times \mathrm{n}}$ are the model coefficient functions — A is the state transition matrix and C is the output matrix. A state space representation $\mathscr{B}(A,C)$ of the model $\mathscr{B}$ is not unique due to a change of basis, i.e.,

$$\mathscr{B} = \mathscr{B}(A,C) = \mathscr{B}(\widehat{A},\widehat{C}),$$

where, for all t

$$\widehat{A}(t) = V(t+1)A(t)V^{-1}(t) \quad \text{and} \quad \widehat{C}(t) = C(t)V^{-1}(t), \quad (2)$$

with a nonsingular matrix $V(t) \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$.

In this paper, we consider the subclass of autonomous linear time-varying systems, for which the coefficient functions A and C are periodic with period P

$$A(t) = A(t+kP) \quad \text{and} \quad C(t) = C(t+kP), \quad \text{for all } t \text{ and } k.$$

Such systems are called linear periodically time-varying and are parameterized in state space by two matrix sequences

$$(A_1, \ldots, A_P) \quad \text{and} \quad (C_1, \ldots, C_P),$$

such that

$$A(t) = A_{(t-1) \bmod P + 1} \quad \text{and} \quad C(t) = C_{(t-1) \bmod P + 1}.$$

The nonuniqueness of the coefficients functions (A,C) of a periodic time-varying system's state space representation is given by (2). In order to preserve the periodicity of the coefficient functions, however, we restrict our attention to state transformations V to periodic, i.e., $V(t) = V_{(t-1) \bmod P + 1}$, for some

$$(V_1, \ldots, V_P), \quad \text{where } V_i \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}} \text{ and } \det(V_i) \neq 0.$$

The class of autonomous linear periodically time-varying systems with order at most n and period P is denoted by $\mathscr{L}_{0,\mathrm{n},P}$. (The zero subscript index stands for zero inputs.)

Problem 1 (*Realization of an autonomous linear periodically time-varying system*) *Given a trajectory*

$$y = \big(y(1),\ldots,y(T)\big),$$

of an autonomous linear periodically time-varying system $\mathscr{B}$, the period P of $\mathscr{B}$, and the state dimension $\mathtt{n}$ of $\mathscr{B}$, find a state space representation $\mathscr{B}(\widehat{A},\widehat{C})$ of the system $\mathscr{B}$, i.e.,

$$\text{find} \quad \widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P} \quad \text{such that} \quad y \in \widehat{\mathscr{B}}.$$

The assumption that the order $\mathtt{n}$ of $\mathscr{B}$ is given can be relaxed, see Note 2.

Notation

- $\mathscr{B}(A,C)$, defined in (1), is a linear autonomous periodically time-varying system with state space parameters (A,C). When A and C are constant matrices (rather than matrix sequences) the system is linear time-invariant.
- For a vector time series $y = \big(y(1),\ldots,y(T)\big)$, $y(t) \in \mathbb{R}^{\mathtt{p}}$ we define the $\mathtt{p}L \times (T-L+1)$ block-Hankel matrix with L, $1 \le L \le T$ block-rows

$$\mathscr{H}_L(y) := \begin{bmatrix} y(1) & y(2) & y(3) & \cdots & y(T-L+1) \\ y(2) & y(3) & \cdot^{\cdot^{\cdot}} & & y(T-L+2) \\ y(3) & \cdot^{\cdot^{\cdot}} & & & \vdots \\ \vdots & & & & \\ y(L) & y(L+1) & \cdots & & y(T) \end{bmatrix}.$$

- The extended observability matrix of a linear periodically time-varying system with a state space representation $\mathscr{B}(A,C)$ is

$$\mathscr{O}_L(A,C) := \begin{bmatrix} C(1) \\ C(2)A(1) \\ C(3)A(2)A(1) \\ \vdots \\ C(L)A(L-1)A(L-2)\cdots A(1) \end{bmatrix}.$$

- The "lifting" operator

$$\text{lift}_P(y) = \big(y'(1),\ldots,y'(T')\big)$$
$$= \left(\begin{bmatrix} y(1) \\ \vdots \\ y(P) \end{bmatrix}, \begin{bmatrix} y(P+1) \\ \vdots \\ y(2P) \end{bmatrix}, \ldots, \begin{bmatrix} y((T'-1)P) \\ \vdots \\ y(T'P) \end{bmatrix} \right), \quad (3)$$

with $T' := \lfloor \frac{T}{P} \rfloor$, ($\lfloor a \rfloor$ is the largest integer smaller than a) sub-samples the $\mathtt{p}$-dimensional vector sequence y at a period P starting from the 1st, 2nd, ..., Pth sample and stacks the resulting P sequences in an augmented $\mathtt{p}' := \mathtt{p}P$-dimensional vector sequence y'—the lifted sequence. Applied on a system $\mathscr{B}$, the operator lift$_P$ acts on all trajectories of the system.

3 Realization of the lifted system

As shown in (Bittanti and Colaneri, 2008, Section 6.2.3), the lifted system lift$_P\big(\mathscr{B}(A,C)\big)$ admits an $\mathtt{n}$th order linear time-invariant representation

$$\mathscr{B}(\widehat{\Phi},\widehat{\Psi}) = \text{lift}_P\big(\mathscr{B}(A,C)\big), \ \text{with} \ \widehat{\Phi} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \ \text{and} \ \widehat{\Psi} \in \mathbb{R}^{\mathtt{p}'\times\mathtt{n}}.$$

The problem of obtaining the parameters $\widehat{\Phi}$ and $\widehat{\Psi}$ from the lifted trajectory y' of the periodically time-varying system is a classical linear time-invariant realization problem. We use Kung's method Kung (1978), which is based on the Hankel matrix $\mathscr{H}_L(y')$. The number of block-rows L must be such that both the number of rows and the number of columns of $\mathscr{H}_L(y')$ are greater than $\mathtt{n}$.

Note 1 (On the choice of the parameter L) *From the point of view of minimizing the computational cost, L is chosen as small as possible, i.e.,*

$$L = L_{\min} := \left\lceil \frac{\mathtt{n}+1}{\mathtt{p}'} \right\rceil,$$

($\lceil a \rceil$ is the smallest integer larger than a). In the presence of noise, however, the accuracy of Kung's algorithm is improved by increasing L. In Kumaresan and Tufts (1982), it is shown that best approximation is achieved by choosing L so that the Hankel matrix $\mathscr{H}_L(y')$ is close to square, i.e.,

$$L = L_{sq} := \left\lceil \frac{T'+1}{\mathtt{p}'+1} \right\rceil.$$

Let

$$\mathscr{H}_L(y') = \mathscr{O}\mathscr{C}, \quad \text{where} \quad \begin{array}{l} \mathscr{O} \in \mathbb{R}^{L\mathtt{p}'\times\mathtt{n}} \\ \mathscr{C} \in \mathbb{R}^{\mathtt{n}\times(T'-L)} \end{array} \quad (4)$$

be a rank revealing factorization of the block-Hankel matrix $\mathscr{H}_L(y')$. Such a factorization can be obtained, for example, from the singular value decomposition

$$\mathscr{H}_L(y') = USV'^{\top} = \underbrace{U\sqrt{S}}_{\mathscr{O}} \underbrace{\sqrt{S}V'^{\top}}_{\mathscr{C}}, \quad (5)$$

where

$$\sqrt{S} := \text{diag}(\sqrt{s_1},\ldots,\sqrt{s_n}).$$

The parameter $\widehat{\Psi}$ is set equal to the first $\mathtt{p}'$ rows of the matrix $\mathscr{O}$ and $\widehat{\Phi}$ is computed from the shift equation

$$\widehat{\Phi}\underline{\mathscr{C}} = \overline{\mathscr{C}}, \quad (6)$$

where $\underline{\mathscr{C}}$ is $\mathscr{C}$ with the last column removed and $\overline{\mathscr{C}}$ is $\mathscr{C}$ with the first column removed.

Note 2 (Unknown order $\mathtt{n}$) *If the order of the linear periodically time-varying system is not given a priori, it can be determined from the rank of $\mathscr{H}_L\big(\text{lift}_P(y)\big)$.*

Note 3 (Inexact data and model reduction) *Truncation of the singular value decomposition (5) is a method to perform (unstructured) low-rank approximation, which has the system*

theoretic interpretation of identifying reduced order model. In the case of perturbation of exact data by noise, truncation of the singular value decomposition to the order of the true system has the effect of signal de-noising.

The $\mathscr{C}$ factor of the factorization (4) has the form

$$
\begin{aligned}
\mathscr{C} &= \begin{bmatrix} \widehat{x}_{\text{ini}} & \widehat{\Phi}\widehat{x}_{\text{ini}} & \cdots & \widehat{\Phi}^{T'-L+1}\widehat{x}_{\text{ini}} \end{bmatrix} \\
&= \begin{bmatrix} \widehat{x}'(1) & \widehat{x}'(2) & \cdots & \widehat{x}'(T'-L) \end{bmatrix} \\
&=: \begin{bmatrix} \widehat{X}'_1 & \widehat{x}'(T'-L) \end{bmatrix},
\end{aligned}
\tag{7}
$$

where $\widehat{x}_{\text{ini}}$ is the initial condition and $\widehat{x}'(1)$, $\widehat{x}'(2)$, … is the state sequence of the linear time-invariant model $\mathscr{B}(\widehat{\Phi},\widehat{\Psi})$. The initial condition $\widehat{x}_{\text{ini}}$ can be obtained directly from $\mathscr{C}$ or it can be re-estimated back from the data by solving the overdetermined system of linear equations

$$
y' = \mathscr{O}_{T'}(\widehat{\Phi},\widehat{\Psi})\widehat{x}_{\text{ini}},
\tag{8}
$$

and defining

$$
\widehat{x}'(t') := \Phi^{t'-1}\widehat{x}_{\text{ini}}, \quad \text{for } t' = 1,2,\dots
$$

In the numerical examples of Section 6, we set the initial condition $\widehat{x}_{\text{ini}}$ equal to the first column of $\mathscr{C}$.

Note 4 (Inexact data and model reduction) *In the case of noisy data or a true system that is not in the model class (see Note 3), (8) generically has no exact solution. Then, the least-squares approximate solution can be used as a means of estimating the initial condition from (8).*

4 Computation of the linear time-varying system's parameters

4.1 Indirect method

Define the matrices

$$
\widehat{X}'_i := V_i \begin{bmatrix} x(i) & x(i+P) & x(i+2P) & \cdots & x(i+(T'-L-1)P) \end{bmatrix},
$$
$$
\text{for } i = 1,\dots,P,
$$

constructed from the state sequence $(x(1),x(2),\dots)$ in a state-space basis, defined by V_i. The derivation of $\widehat{X}'_1$ is a by-product of the realization of the lifted system $\mathscr{B}(\widehat{\Phi},\widehat{\Psi})$, see (7). The shifted state sequences $\widehat{X}'_2,\dots,\widehat{X}'_P$ can also be computed from (4) by using the i-steps shifting data $(y(i),\dots,y(T))$ instead of $(y(1),\dots,y(T))$. Note that the computation of $\widehat{X}'_i$ through (4) results in general in a basis V_i that is different from V_j, for $i \neq j$.

The model parameters $(\widehat{A},\widehat{C})$ are computed from the equations

$$
\begin{bmatrix} \widehat{X}'_{i+1} \\ Y_i \end{bmatrix} = \begin{bmatrix} \widehat{A}_i \\ \widehat{C}_i \end{bmatrix} \widehat{X}'_i, \qquad \text{for } i = 1,\dots,P,
\tag{9}
$$

where

$$
Y_i := \begin{bmatrix} y(i) & y(i+P) & y(i+2P) & \cdots & y(i+(T'-L-1)P) \end{bmatrix}.
$$

The matrix $\widehat{X}'_{P+1}$ is obtained from $\widehat{X}'_1$ by pre-multiplication with Φ (i.e., shift with P steps forward)

$$
\widehat{X}'_{P+1} := \Phi\widehat{X}'_1.
$$

This guarantees that $\widehat{X}'_{P+1}$ is in the same basis as $\widehat{X}'_1$, which implies that $V_1 = V_{P+1}$.

Algorithm 1 summarizes the method for realization of linear periodically time-varying systems, described above.

Algorithm 1 Indirect algorithm for linear periodically time-varying system realization. # of operations
Input: Sequence $y \in (\mathbb{R}^{\text{p}})^T$ and natural numbers P and n.
(1) *lifting:* (3)   0
(2) *realization of the lifted system:* $(y',\text{n}) \mapsto \mathscr{B}(\widehat{\Phi},\widehat{\Psi})$ ((4) and (6)) $O((L\text{p}')^2 T')$
(3) *state estimation:* compute the state sequence matrices $\widehat{X}'_1,\dots,\widehat{X}'_P$ and define $\widehat{X}'_{P+1} := \widehat{\Phi}\widehat{X}'_1$ $O(P(L\text{p}')^2 T')$
(4) *parameter estimation:* solve the systems (9) $O(P(\text{n}+\text{p}')^2)$
Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.   overall cost: $O(L^2\text{p}^2 P^2 T)$

4.2 Direct method

The most expensive step of Algorithm 1 is the computation of the shifted state sequences $\widehat{X}'_i$, which requires P factorizations of block-Hankel matrices. As proven in the following proposition, the parameters $(\widehat{A},\widehat{C})$ of the linear periodically time-varying system can be obtained directly from the parameters $(\widehat{\Phi},\widehat{\Psi})$ of the linear time-invariant system, without extra computation.

Proposition 1 *The linear periodically time-varying system $\mathscr{B}(\widehat{A},\widehat{C})$, with parameters*

$$
\widehat{A}_1 := I_n, \ \dots, \ \widehat{A}_{P-1} := I_n, \ \widehat{A}_P := \widehat{\Phi}
\tag{10}
$$
$$
\widehat{\Psi} =: \text{col}(\widehat{C}_1,\dots,\widehat{C}_P), \quad \text{where } \widehat{C}_i \in \mathbb{R}^{\text{p}\times\text{n}},
\tag{11}
$$

is equivalent to the linear time-invariant system $\mathscr{B}(\widehat{\Phi},\widehat{\Psi})$, i.e., $\mathscr{B}(\widehat{\Phi},\widehat{\Psi}) = \text{lift}_P(\mathscr{B}(\widehat{A},\widehat{C}))$.

PROOF. We have to show that a response y of the system $\text{lift}_P(\mathscr{B}(\widehat{A},\widehat{C}))$ is also a response of the linear time-invariant system $\mathscr{B}(\widehat{\Phi},\widehat{\Psi})$. Let x_{ini} be the initial condition of the linear periodically time-varying system $\mathscr{B}(\widehat{A},\widehat{C})$ that generates y. We have

$$
y(1) = \widehat{C}_1 x_{\text{ini}}, \ \dots, y(P) = \widehat{C}_P x_{\text{ini}}
$$
$$
y(P+1) = \widehat{C}_1\widehat{\Phi}x_{\text{ini}}, \ \dots, y(2P) = \widehat{C}_P\widehat{\Phi}x_{\text{ini}}
$$
$$
\vdots
$$
$$
y(t'P+1) = \widehat{C}_1\widehat{\Phi}^{t'}x_{\text{ini}}, \ \dots, y(2P) = \widehat{C}_P\widehat{\Phi}^{t'}x_{\text{ini}}.
$$

On the other hand, the response of the linear time-invariant system $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$ to the initial condition x_{ini} is

$$y'(t') = \widehat{\Psi}\widehat{\Phi}^{t'} x_{\text{ini}} = \begin{bmatrix} \widehat{C}_1 \\ \vdots \\ \widehat{C}_P \end{bmatrix} \widehat{\Phi}^{t'} x_{\text{ini}}, \quad \text{for } t' = 0, \ldots, T'-1.$$
$$(12)$$

It follows that $\text{lift}_P(y) = y'$.

Algorithm 2 summarizes the direct method for realization of linear periodically time-varying systems.

Algorithm 2 Direct algorithm for linear periodically time-varying system realization.　　　　# of operations

Input: Sequence $y \in (\mathbb{R}^p)^T$ and natural numbers P and n.
 (1) *lifting:* (3)　　　　　　　　　　　　　　　　0
 (2) *modeling:* $(y', \text{n}) \mapsto \mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$
　　　(e.g., Kung's algorithm: (4) and (6))　　$O((Lp')^2 T')$
 (3) define $\widehat{A}$ and $\widehat{C}$ via (11)　　　　　　　　0
Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.　　　　overall cost: $O(L^2 p^2 PT)$

4.3 Modification of the direct method

Consider the "transposed" lifted sequence

$$y'^\top := \left(y'^\top(1), \ldots, y'^\top(T') \right), \qquad y'^\top(t) \in \mathbb{R}^{1 \times p'}$$

and the associated $L \times p'(T'-L+1)$ block-Hankel matrix

$$\mathscr{H}_L(y'^\top) := \begin{bmatrix} y'^\top(1) & y'^\top(2) & y'^\top(3) & \cdots & y'^\top(T'-L+1) \\ y'^\top(2) & y'^\top(3) & \cdot^{\cdot^\cdot} & & y'^\top(T'-L+2) \\ y'^\top(3) & \cdot^{\cdot^\cdot} & & & \vdots \\ \vdots & & & & \\ y'^\top(L) & y'^\top(L+1) & \cdots & & y'^\top(T') \end{bmatrix}.$$
$$(13)$$

The parameter L satisfies the constraints $L > \text{n}$ and $p(T'-L-1) > \text{n}$. As before, for best approximation accuracy, L is selected to make $\mathscr{H}_L(y'^\top)$ as square as possible. For minimal computational cost, L is chosen as small as possible, which in the case of (13) is $\text{n}+1$.

Since y is a trajectory of the linear periodically time-varying system $\mathscr{B}(A,C)$, we have that

$$\mathscr{H}_L(y'^\top) = \mathscr{O}_L(\widehat{\Phi}^\top, x_{\text{ini}}^\top) \cdot \mathscr{O}_{T'-L+1}^\top(\widehat{\Phi}, \widehat{\Psi}).$$

Therefore, the parameters $(\widehat{\Psi}, \widehat{\Phi})$ of the lifted system can be identified from the rank revealing factorization

$$\mathscr{H}_L(y'^\top) = USV^\top = \underbrace{U\sqrt{S}}_{\mathscr{O}} \underbrace{\sqrt{S}V^\top}_{\mathscr{C}}.$$
$$(14)$$

The initial condition x_{ini} is the transposed first row of the $\mathscr{O}$ factor, Ψ is the transposed first $\text{n} \times \text{p}$ block element of $\mathscr{C}$, and $\widehat{\Phi}^\top$ is a solution of the shift equation

$$\underline{\mathscr{O}}\widehat{\Phi}^\top = \overline{\mathscr{O}},$$
$$(15)$$

where $\underline{\mathscr{O}}$ is $\mathscr{O}$ with the last row removed and $\overline{\mathscr{O}}$ is $\mathscr{O}$ with the first row removed.

The resulting identification method is summarized in Algorithm 3.

Algorithm 3 Modified direct algorithm for linear periodically time-varying realization.　　　　# of operations

Input: Sequence $y \in (\mathbb{R}^p)^T$ and natural numbers P and n.
 (1) *lifting:* (3)　　　　　　　　　　　　　　　0
 (2) *modeling:* $(y', \text{n}) \mapsto \mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$　　((14) and (15))
　　　　　　　　　　　　　　　　　　$O(L^2 p' T')$
 (3) define $\widehat{A}$ and $\widehat{C}$ via (11)　　　　　　　　0
Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.　　　　overall cost: $O(L^2 pT)$

Note that the computational cost of Algorithm 3 is independent of the period P and is linear in the number of outputs p. This is a significant improvement over Algorithm 2. In addition, as discussed in the next section, using the matrix $\mathscr{H}_L(y'^\top)$ instead of $\mathscr{H}_L(y)$ has an important advantage in the case of optimal approximate identification.

5 Maximum likelihood identification

As commented in Notes 3 and 4, the realization algorithms 1–3 can be used in the case of noisy data as estimation methods. Using instrumental variables, the basic algorithms presented can be extended to different noise assumptions, resulting in a class of the non-iterative identification methods, such as the MOESP methods Verhaegen and Dewilde (1992). Despite many advantages, however, non-iterative methods do not estimate optimal (in an a priori specified sense) models. Therefore, the problem of iteratively refining the model computed by subspace methods using optimization-based methods is considered next.

5.1 Noise assumptions

Assume that the data is generated in the output error setup:

$$y = \overline{y} + \widetilde{y}, \qquad \text{where} \quad \overline{y} \in \overline{\mathscr{B}} \in \mathscr{L}_{0,\text{n},P}$$
$$\text{and } \widetilde{y} \text{ is zero mean white Gaussian} \qquad (16)$$
$$\text{process with covariance matrix } \xi^2 I_{\text{p}}.$$

The "true value" $\overline{y}$ of the data y is generated by a linear periodically time-varying system $\mathscr{B}(\bar{A}, \bar{C})$, refered to as the "true system". Persistency of excitation of the lifted true trajectory $\overline{y}'$

$$\text{rank}\left(\mathscr{H}_{L_{\min}}(\overline{y}') \right) = \text{n} \qquad (17)$$

is required for identifiability of the data generating system. Note that (17) imposes implicitly a condition on the initial state $\overline{x}_{\text{ini}}$ as well as on the true system $\overline{\mathscr{B}}$.

Our aim is to estimate the true linear periodically time-varying system $\mathscr{B}(\bar{A}, \bar{C})$ from the data y and the prior knowledge that the true system belongs to the model class $\mathscr{L}_{0,\mathtt{n},P}$. The log likelihood function for the data generating model (16) is

$$L(\widehat{\mathscr{B}}, \widehat{y}) = \begin{cases} \text{const} - \frac{1}{2\xi^2} \|y - \widehat{y}\|_2^2 & \text{if } \widehat{y} \in \widehat{\mathscr{B}} \\ -\infty & \text{otherwise} \end{cases}$$

The maximization of L leads to the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \text{over } \widehat{y} \text{ and } \widehat{\mathscr{B}} \quad \|y - \widehat{y}\|_2 \\ \text{subject to} \quad & \widehat{y} \in \widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P}. \end{aligned} \tag{18}$$

Problem 2 *(Maximum likelihood identification of an autonomous linear periodically time-varying system) Given a trajectory*
$$y = (y(1), \ldots, y(T)),$$
and a model class $\mathscr{L}_{0,\mathtt{n},P}$, specified by natural numbers $\mathtt{n}$ and P, find a minimizer $\widehat{\mathscr{B}}$ of (18)

For noisy data, the identified model has generically order $\mathtt{n}$. If the data is exact and is generated by a linear periodically time-varying system of order less than $\mathtt{n}$, a nonminimal exact model exists. This case is easy to detect (e.g., by checking the rank of the Hankel matrix $\mathscr{H}_{\mathtt{n}+1}(\text{lift}_P(\widehat{y}^\top))$) and does not require optimization.

In Problem 2, no assumption is made about the initial conditions from which the data y is generated. Consequently, in the optimization problem (18), the initial conditions that generate $\widehat{y}$ are unconstrained.

5.2 Equivalence to structured total least squares

As shown in (Markovsky *et al.*, 2005, Section III, Theorem 1),

$$\begin{aligned} \widehat{y} \in \widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P} \quad &\Longleftrightarrow \quad \text{rank}\left(\mathscr{H}_{\mathtt{n}+1}(\text{lift}_P(\widehat{y}^\top))\right) \le \mathtt{n} \\ \text{and} \quad &\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \notin \text{leftker}\left(\mathscr{H}_{\mathtt{n}+1}(\text{lift}_P(\widehat{y}^\top))\right). \end{aligned}$$

so that, the optimization problem (18) is equivalent to a Hankel structured total least squares approximation problem.

Proposition 2 *(Optimal identification of linear periodically time-varying system via structured low-rank approximation) Problem 2 is equivalent to the structured total least squares approximation problem*

$$\begin{aligned} \text{minimize} \quad & \text{over } \widehat{y} \quad \|y - \widehat{y}\|_2 \\ \text{subject to} \quad & \text{rank}\left(\mathscr{H}_{\mathtt{n}+1}(\text{lift}_P(\widehat{y}^\top))\right) \le \mathtt{n} \\ \text{and} \quad & \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \notin \text{leftker}\left(\mathscr{H}_{\mathtt{n}+1}(\text{lift}_P(\widehat{y}^\top))\right). \end{aligned} \tag{SLRA}$$

For the solution of the structured low-rank approximation problems (SLRA) we use the method of Usevich and Markovsky (2013). It is based on the kernel representations of the rank constraint

$$\begin{aligned} \text{rank}\left(\mathscr{H}_{\mathtt{n}+1}(\text{lift}_P(\widehat{y}^\top))\right) \le \mathtt{n} \quad &\Longleftrightarrow \\ \text{there is an } R^{1 \times (\mathtt{n}+1)}, \text{ such that} \\ R\mathscr{H}_{\mathtt{n}+1}(\text{lift}_P(\widehat{y}^\top)) = 0 \text{ and } R_{\mathtt{n}+1} = 1. \end{aligned}$$

In the method of Usevich and Markovsky (2013), the variable projection approach is used Golub and Pereyra (2003), i.e., the optimization variable $\widehat{y}$ is eliminated for a fixed R by analytically minimizing over it (a linear least norm problem). The resulting nonlinear least squares problem for R is solved by local optimization methods.

The solution $\widehat{y}$ of the structured low-rank approximation is by construction an exact trajectory of a system in the model class $\mathscr{L}_{0,\mathtt{n},P}$. Therefore, the remaining problem of finding the model for $\widehat{y}$ (which is the optimal approximate model for y) is an exact identification problem and can be solved by Algorithm 3. The structured low-rank approximation methods of Usevich and Markovsky (2013), however, returns as a byproduct the kernel matrix R. Therefore, a rank revealing factorization (14) can be computed without using the computationally more expensive singular value decomposition.

The orthogonal complement $R^\perp$ of R is equal to the left factor $\mathscr{O}$ in (5). Knowledge of $\mathscr{O}$ is sufficient to determine the parameters $(\widehat{\Phi}, \widehat{\Psi})$ of the lifted system. The resulting optimal identification method is summarized in Algorithm 3.

Algorithm 4 Algorithm for optimal linear periodically time-varying system identification. # of operations

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Input: Sequence $y \in (\mathbb{R}^\mathtt{p})^T$ and natural numbers P and $\mathtt{n}$.
  (1) *lifting:* (3)   0
  (2) *modeling:* $(y', \mathtt{n}) \mapsto (R, \widehat{y}')$ (SLRA)
                        $O((\mathtt{n}+1)^3 \mathtt{p}' T')$ per iteration
  (3) Compute $\mathscr{O} = R^\perp$ and define $\widehat{A}$ and $\widehat{C}$ via (11).
                        $O((\mathtt{n}+1)^3)$
Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.
          overall cost (for K iterations): $O((\mathtt{n}+1)^3 \mathtt{p} TK)$

The methods in the paper are implemented in MATLAB and are available in the `ident` directory of the structured low-rank approximation package Markovsky and Usevich (2014):

    `http://slra.github.io/`

The simulation results presented in the following section can be reproduced with the m-file `pltv_all_examples`.

5.3 Properties of an estimator

There are no linear or nonlinear transformations involved in the lifting $\text{lift}_P(y)$ of the data and the transition from the identified parameters $(\widehat{\Phi}, \widehat{\Psi})$ of the lifted system to the parameters $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$ of the equivalent linear periodically time-varying system (see, Proposition 1). Therefore, the properties of an estimator of the linear periodically time-varying system are inherited from the corresponding properties of the estimator, used for the identification of the lifted system.

In particular, the properties of the maximum likelihood estimator in the linear time-invariant case are well known. In the autonomous case, the problem is extensively studied in the signal processing literature, where it is better known as "sum-of-damped exponentials estimation" and "linear prediction". In this case, the maximum likelihood estimator is, in general, not consistent. Consistency can be recovered in the special case of marginally stable system (undamped exponentials), see Favaro and Picci (2012), or by using data of repeated experiments.

Note 5 (Well damped systems) *The trajectories of a well damped autonomous system are quickly decaying. In the presence of noise, this effectively limits the number of samples that can be used in an identification experiment. The issue of the short response with sufficiently high signal-to-noise ratio is intrinsic to the identification problem of well damped autonomous system and is reflected in the lack of consistent estimation methods.*

6 Numerical examples

The estimation accuracy of Algorithms 1–4, measured by the prediction error, is compared on a test example from Allen *et al.* (2011). In Section 6.2, the computational advantages of the modified method (Algorithm 3) over the classical method (Algorithm 1) is illustrated on a marginally stable linear periodically time-varying system. Identification with missing data is shown in Section 6.3 and statistical properties of the maximum likelihood estimator (Algorithm 4) are shown in Section 6.4. Finally, in Section 6.4 we show a simulation example of a sixth order system with realistic values of the model parameters.

6.1 Comparison of the methods on Mathieu oscillator

In all examples, the data is generated according to the output error model (16). In this subsection, the true data generating system is Mathieu oscillator—a spring-mass-damper system with time-periodic spring stiffness. A state-space representation of Mathieu oscillator is

$$\bar{A}_\tau = \begin{bmatrix} 0 & 1 \\ \bar{a}_1 & \bar{a}_{2,\tau} \end{bmatrix}, \quad \bar{C}_\tau = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \text{for } \tau = 1,\dots,P,$$

where, in the particular simulation example shown, the parameters are

$$\bar{a}_1 = -0.9 \quad \text{and} \quad \bar{a}_{2,\tau} = -\big(0.1 + 0.4\cos(2\pi\tau/P)\big).$$

The periods length is $P = 3$ and the data y consists of $T' = 20$ periods.

Algorithms 1–4 are applied on the first 3/4 of the simulated data (identification data) and the obtained models are evaluated in terms of the relative prediction error

$$e = \frac{\|\bar{y}_{\text{val}} - \widehat{y}_{\text{val}}\|_2}{\|\bar{y}_{\text{val}} - \text{mean}(\bar{y}_{\text{val}})\|_2} \tag{19}$$

on the remaining 1/4 of the data (validation data). The reported results are averaged over 100 noise realizations.

The identification experiment is repeated for a range of noise variances ξ^2. Table 1 shows the averaged relative approximation errors e for all methods and all noise variances. Among the subspace methods, best estimation accuracy achieves Algorithm 3. Used as an initial approximation for the maximum-likelihood algorithm, the estimate of Algorithm 3 is further improved by Algorithm 4. Figure 2 shows the bias and variance components of the error.

6.2 Computational efficiency on large examples

In this subsection, we illustrate the computational advantages of the proposed in the paper Algorithm 3 over the classical Algorithm 1. In order to apply the methods on an example with large number of samples, the true data is generated by a marginally stable autonomous linear periodically time-varying system, i.e., the eigenvalues $\bar{z}_1, \dots, \bar{z}_n$ of $\bar{A}_1\bar{A}_2\dots\bar{A}_P$ are chosen on the unit circle (observing the complex conjugate symmetry) and all have multiplicity one.

Define

$$\bar{\theta}(z) := (z - \bar{z}_1)\cdots(z - \bar{z}_n) = z^n + \bar{\theta}_1 z^{n-1} + \cdots + \bar{\theta}_n$$

and let

$$\bar{\theta}^\top := \begin{bmatrix} \bar{\theta}_1 & \cdots & \bar{\theta}_n \end{bmatrix}$$

be the true system's parameter vector. Similarly, let $\widehat{\theta}$ be the identified system's parameter vector. The estimation accuracy is measured by the relative parameter error

$$e = \frac{\|\bar{\theta} - \widehat{\theta}\|_2}{\|\bar{\theta}\|_2}, \tag{20}$$

averaged over 100 Monte Carlo repetitions of the identification with different noise realizations.

The reported results of computation error and parameter error are shown for randomly generated examples with $T = 10000$ data points and period P ranging from 10 to 1000. The average computational times of the subspace algorithms are shown in Table 2 and the corresponding approximation errors in Table 3. Algorithm 3 is several orders of magnitude faster than Algorithm 1.

6.3 Missing data

The possibility to solve linear periodically time-varying identification problems with missing data, using the method of Markovsky and Usevich (2013), is illustrated in this subsection on the example of Section 6.1 (Mathieu oscillator). The signal-to-noise ratio is 10 and a fraction of the output samples are missing in a random pattern. Algorithm 4 is applied on the noisy incomplete data and the identified model is validated by computing the relative estimation error of the missing samples

$$e = \frac{\|\bar{y}_{\text{missing}} - \widehat{y}_{\text{missing}}\|_2}{\|\bar{y}_{\text{missing}} - \text{mean}(\bar{y}_{\text{missing}})\|_2}. \tag{21}$$

The results reported in Table 4 are averaged over 100 different random patterns of the missing values.

	Inf	12	6	4	3	2.4	2
Alg. 1	0.0000	0.0794	0.1625	0.2316	0.3137	0.4071	0.4635
Alg. 2	0.0000	0.0795	0.1600	0.2253	0.3128	0.3981	0.4585
Alg. 3	0.0000	0.0788	0.1550	0.2244	0.3058	0.3845	0.4525
Alg. 4	0.0000	0.0736	0.1473	0.2123	0.2738	0.3577	0.4315

Table 1
Relative prediction errors (19) in identifying Mathieu oscillator with signal-to-noise ratios varying from ∞ (exact data) to 2.
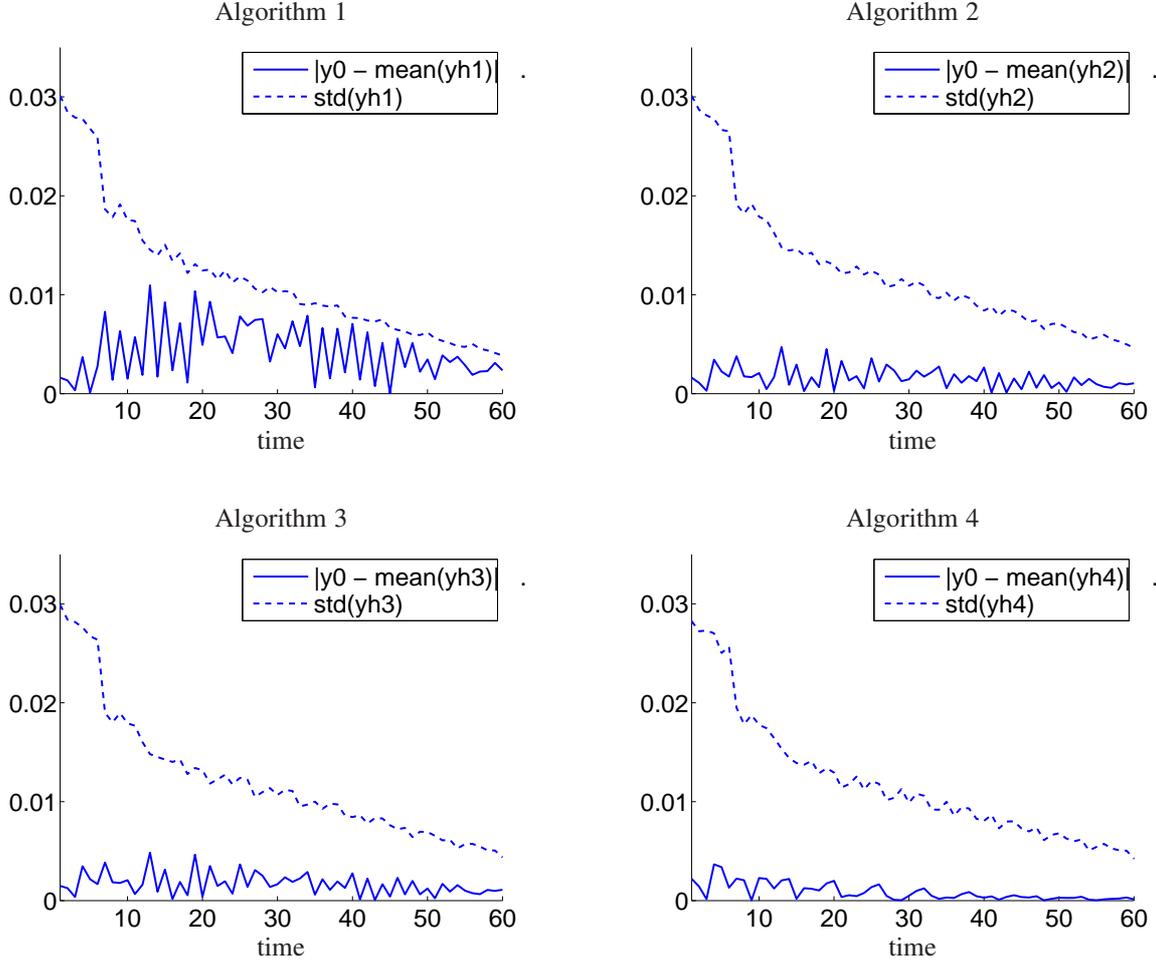
Fig. 2. Bias $|\bar{y} - \text{mean}(\hat{y})|$ and standard deviation $\text{std}(\hat{y})$ of the modeled responses.

	10	100	200	500	750	1000
Alg. 1	3.0684	0.2137	0.1870	0.1744	0.1978	0.2350
Alg. 3	0.2998	0.0021	0.0013	0.0007	0.0007	0.0007

Table 2
Computation times in seconds on problems with $T = 10000$ samples and period lengths from 10 to 1000.

	10	100	200	500	750	1000
Alg. 1	0.0000	0.0999	0.0081	0.0160	0.5425	0.5614
Alg. 3	0.0000	0.0357	0.0027	0.0079	0.6457	0.6067

Table 3
Average relative error (20) on problems with $T = 10000$ samples and period lengths from 10 to 1000.

	5	10	15	20	25	30	35
Alg. 4	0.0958	0.1165	0.1825	0.2826	0.3067	0.4272	0.6550

Table 4
Relative error in estimation of the missing data (21) as a function of the percentage of the missing data.

6.4 Confidence ellipsoids

In this subsection, we illustrate the consistency of the maximum likelihood estimator in the case of a marginally stable system. We also show the corresponding confidence bound for the estimated parameters. The simulation setup is the same as in Section 6.2. The reported results are generated by a system with period length $P = 100$ and number of samples T from 1000 to 5000. For each sample size, the identification is repeated $N = 100$ times with independent noise realizations (but the true system remains fixed). The parameter estimation error (20) is shown in Figure 3 as a function of the sample size T. Figure 4 shows the true parameters $\bar{\theta}_1, \bar{\theta}_2$ (red cross), the 100 estimates $\bar{\theta}_1^k, \bar{\theta}_2^k$ for $T = 5000$ (blue dots), and the 95% confidence ellipsoid, computed from the covariance matrix of $\widehat{\theta}$, and translated to $\bar{\theta}$.
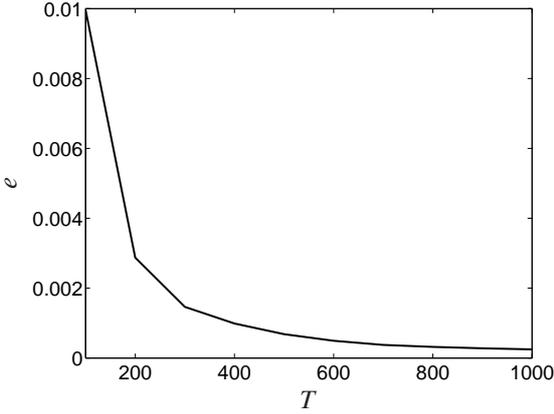
Fig. 3. Estimation error e as a function of the sample size T.
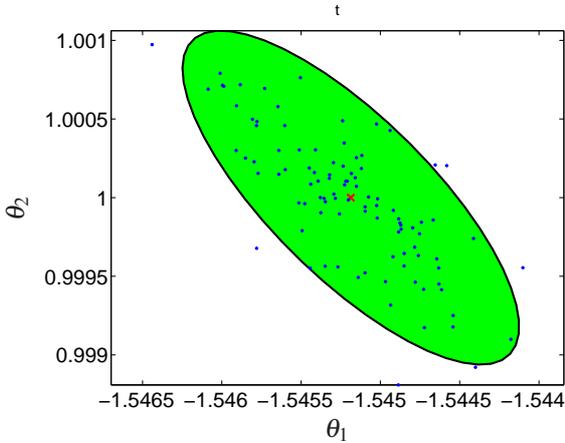
Fig. 4. 95% confidence ellipsoid.

6.5 Realistic 6th order example

The example presented in this section is a 6th order lowly damped periodically time-varying system $\mathscr{B}(A, C)$, with state space parameters

$$A(t) = A_{\mathrm{lti}}\big(1 + \alpha \sin(2\pi f t)\big)$$
$$C(t) = C_{\mathrm{lti}}\big(1 + \gamma \sin(2\pi f t)\big).$$

The terms $\alpha \sin(2\pi f t)$ and $\gamma \sin(2\pi f t)$ define the time variation and have parameters $\alpha = 0.1$, $\gamma = 0.15$, and $f = 20$.

$\mathscr{B}(A_{\mathrm{lti}}, C_{\mathrm{lti}})$ is a 6th order autonomous linear time-invariant system with resonance angular frequencies

$$\bar{\omega}_1 = 2\pi\,80\,\mathrm{rad/s}, \quad \bar{\omega}_2 = 2\pi\,130\,\mathrm{rad/s}, \quad \bar{\omega}_3 = 2\pi\,200\,\mathrm{rad/s}$$

the poles' damping ratios are

$$\zeta_1 = 0.015, \quad \zeta_2 = 0.01, \quad \zeta_3 = 0.02,$$

the transmission zeroes are

$$\omega_{z,1} = 2\pi\,105\,\mathrm{rad/s}, \quad \omega_{z,2} = 2\pi\,165\,\mathrm{rad/s},$$

and zeroes' damping ratios

$$\zeta_{z,1} = 0.001, \quad \zeta_{z,2} = 0.0008.$$

The system is simulated with initial condition

$$x_{\mathrm{ini}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\top}$$

over the interval $[0, 0.6]$, using MATLAB's ordinary differential equation solver `ode45` and is sampled with period 10^{-3}. The identification data are 601 noise perturbed output samples of the continuous-time trajectory. The noise is zero mean independent normally distributed with signal-to-noise ratio 4. The true, noisy, and estimated trajectories are shown in Figure 5.

7 Conclusions

In this paper, we developed realization and maximum likelihood identification algorithms for autonomous linear periodically time-varying systems. The algorithms are based on 1) lifting of the original time series, 2) modeling of the lifted time-series by a linear time-invariant system, 3) the transition from the time-invariant system's parameters to the ones of the periodic time-varying system. It is shown that the derivation of the periodic time-varying system's state space parameters in step 3 can be done without extra computations. Also, in step 2, the realization problem can be solved by a rank revealing factorization of a block Hankel matrix with $\mathtt{n} + 1$ rows, where $\mathtt{n}$ is the order to the system. These facts lead to a new efficient realization algorithm and a maximum likelihood identification algorithm, based on Hankel structured low-rank approximation. Consequently, readily available robust and efficient optimization methods and software can be used for identifying periodic linear time-varying systems.
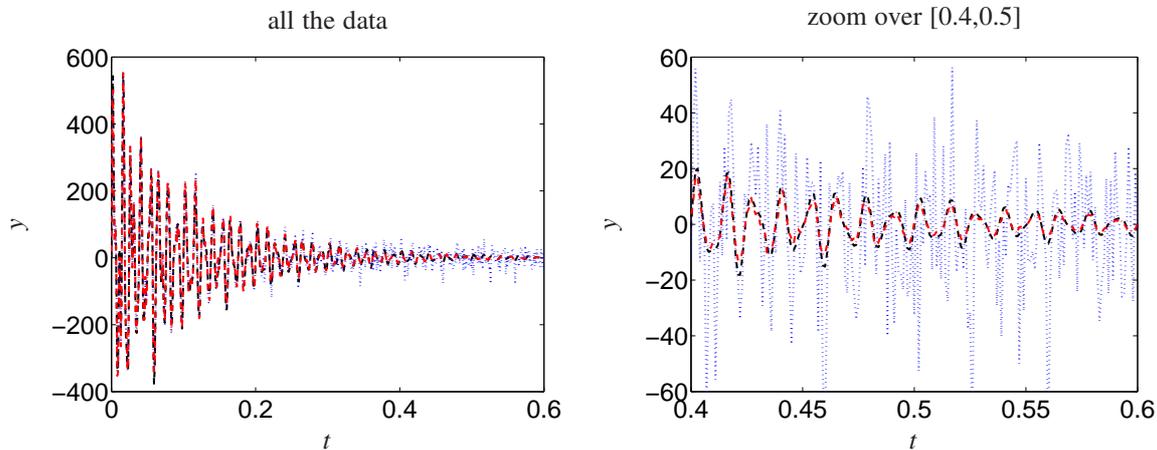
Fig. 5. True (dashed-dotted), noisy (dotted), and estimated (dashed) trajectories in the example of Section 6.5.

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