The Set of Linear Time-Invariant Unfalsified Models with Bounded Complexity is Affine

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***Abstract*—We consider exact system identification in the behavioral setting: given an exact (noise-free) finite time series, find the set of bounded complexity linear time-invariant systems that fit the data exactly. First, we modify the notion of the most powerful unfalsified model for the case of finite data by fixing the number of inputs and minimizing the order. Then, we give necessary and sufficient identifiability conditions, *i.e.*, conditions under which the true data generating system coincides with the most powerful unfalsified model. Finally, we show that the set of bounded complexity exact models is affine: every exact model is a sum of the most powerful unfalsified model and an autonomous model with bounded complexity.**

***Index Terms*—Behaviors, exact system identification, Hankel matrix, most powerful unfalsified model, persistency of excitation.**

I. Introduction

Exact system identification refers to the problem of identifying the true data generating system from an observed trajectory of the system. This problem is formalized by the notion of the most powerful unfalsified model (MPUM), which is the least complicated model explaining the data [1]. The MPUM is originally defined for infinite time series and linear time-invariant (LTI) systems. In this case, the complexity of the model is measured by the ordered pair: (number of inputs, order). Moreover, the MPUM always exists and is unique [1]. Under suitable conditions, called *identifiability conditions*, the MPUM coincides with the data generating system. Therefore, the MPUM is the solution to the exact identification problem. Sufficient identifiability conditions are given in [2].

The problem of exact system identification for LTI behaviors with polynomial-exponential time series has been considered in [3] and later a generalization of the problem to multidimensional behaviors has been investigated in [4]. The problem of minimal partial realization has been considered as an instance of identification problem in the context of exact modeling [5]. Also, the notion of the MPUM led to the development of the subspace identification methods, see for example [6, Chapter 2].

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For finite time series the notion of MPUM needs an adaptation because minimization of model complexity subject to the constraint that the model is exact yields an autonomous system [7]. A possible adaptation allowing identification of open systems, *i.e.*, systems with inputs, is restriction of model's complexity. In the paper, we assume that the number of inputs is a priori given and minimize the order. The resulting MPUM exist and is unique (Lemma 1).

The notion of the MPUM is fundamental also in *data-driven simulation and control*, see, *e.g.*, [8]–[11]. In [11], the authors study properties of the true data generating system and existence of stabilizing controllers that can be inferred from data that is not informative enough for identifiability of the true data generating system. In case of non-informative data, there are infinitely many exact models. We prove that the set of exact models has affine structure. More specifically, any exact model is a sum of the MPUM and an autonomous model, *i.e.*, every exact model must include the MPUM.

The contributions of the paper are: (i) modification of the definition of the MPUM for finite time series; (ii) necessary and sufficient identifiability conditions; (iii) characterization of the set of exact LTI models of bounded complexity for a given finite time series.

In the following section, we recall some notions and concepts from behavioral systems theory that are used in the rest of the paper. (For an overview of behavioral systems theory, we refer the reader to [12].) Then, a modification of the notion of the MPUM for finite time series is given in Section III. In Section IV, we consider the special case of autonomous systems. The general case of open systems is developed in Section V. The results are illustrated on examples in Section VI.

II. Linear Time-Invariant behaviors

A dynamical system (also called a model or just a system) is defined by the triplet $(\mathbb{T}, \mathbb{W}, \mathscr{B})$, where $\mathbb{T} \subseteq \mathbb{R}$ is the time axis, $\mathbb{W} \subseteq \mathbb{R}^q$ is the signal space, and $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the behavior with $\mathbb{W}^{\mathbb{T}}$ the set of all functions $w : \mathbb{T} \to \mathbb{W}$. Note that once we have the behavior, the other two components of the model are already specified. So, in this note we interchangeably use the terms behavior, model, and system. Also, this work considers discrete-time behaviors, so that $\mathbb{T} \subseteq \mathbb{N}$. By $\mathscr{L}^q$, we denote the set of LTI behaviors with q variables that are closed in the topology of point-wise convergence. The sum of two behaviors $\mathscr{B}_i \in \mathscr{L}^q, i = 1, 2$ is a behavior defined as

$$\mathscr{B}_1 + \mathscr{B}_2 := \{w : w = w_1 + w_2, w_1 \in \mathscr{B}_1, w_2 \in \mathscr{B}_2\}.$$

The order (also called state cardinality) and the number of inputs (also called input cardinality) of a system $\mathscr{B} \in \mathscr{L}^q$ are denoted by, respectively, $\mathbf{n}(\mathscr{B})$ and $\mathbf{m}(\mathscr{B})$. The ordered pair

$$\mathbf{c}(\mathscr{B}) := \big(\mathbf{m}(\mathscr{B}), \mathbf{n}(\mathscr{B})\big)$$

is a measure of $\mathscr{B}$'s complexity. The set of LTI systems with q variables and complexity bound (m,n) is denoted by $\mathscr{L}_m^{q,n}$.

Any finite-dimensional LTI system $\mathscr{B} \in \mathscr{L}^q$ admits a *kernel representation* $\mathscr{B} = \{ w \ : \ R(\sigma) w = 0 \}$, where $R \in \mathbb{R}^{g \times q}[\xi]$ is a polynomial matrix and $\sigma : \mathbb{W}^{\mathbb{T}} \to \mathbb{W}^{\mathbb{T}}$ is the *backward shift* operator $(\sigma w)(t) := w(t+1)$. The minimal degree of R in a kernel representation of the system is invariant of the representation. It is called the *lag* of the system and is denoted as $\mathbf{l}(\mathscr{B})$. The lag and the order are related by the inequalities $\mathbf{l}(\mathscr{B}) < \mathbf{n} \leq p\mathbf{l}(\mathscr{B})$, where p is the number of outputs.

The Hankel matrix with $L \in \mathbb{N}$ block-rows for a time series

$$w_{\rm d} := \big(w_{\rm d}(1), w_{\rm d}(2), \dots, w_{\rm d}(T)\big) \in (\mathbb{R}^q)^T$$

is defined as follows:

$$\mathscr{H}_L(w_{\rm d}) := \begin{bmatrix} w_{\rm d}(1) & w_{\rm d}(2) & \cdots & w_{\rm d}(T-L+1) \\ w_{\rm d}(2) & w_{\rm d}(3) & \cdots & w_{\rm d}(T-L+2) \\ \vdots & \vdots & \ddots & \vdots \\ w_{\rm d}(L) & w_{\rm d}(L+1) & \cdots & w_{\rm d}(T) \end{bmatrix}.$$

Throughout the paper, L is a user defined parameter, which has the meaning of time-horizon or window-length. The *persistency of excitation order* of a time series $w_{\rm d} \in (\mathbb{R}^q)^T$ is the maximum L, for which $\mathscr{H}_L(w_{\rm d})$ has full row rank. An upper bound of the persistency of excitation order of $w_{\rm d} \in (\mathbb{R}^q)^T$ is

$$L^* := \left\lceil \frac{T+1}{q+1} \right\rceil.$$

(For $L > L^*$, $\mathscr{H}_L(w_{\rm d})$ has more rows than columns.)

The *restriction* of the behavior $\mathscr{B}$ to the interval $[1,L]$ is defined as follows:

$$\mathscr{B}|_L := \{ w \in (\mathbb{R}^q)^L \mid \text{there is } v \in \mathscr{B}, \text{ such that } w(t) = v(t) \text{ for all } 1 \leq t \leq L \}.$$

Summary of Notation:

$\mathscr{B}$ / $\mathscr{B}\vert_L$	behavior / restriction of behavior to $[1,L]$
$\mathscr{L}^q$	set of all LTI models with q variables
$\mathscr{L}_m^q$	set of models in $\mathscr{L}^q$ with at most m inputs
$\mathscr{L}_m^{q,n}$	set of models in $\mathscr{L}^q$ with complexity bounded by (m,n)
$\mathbf{n}(\mathscr{B})$ / $\mathbf{l}(\mathscr{B})$	order / lag of $\mathscr{B}$
$\mathbf{m}(\mathscr{B})$ / $\mathbf{c}(\mathscr{B})$	number of inputs / complexity of $\mathscr{B}$
span	span of the columns of a matrix or span of a set of vectors
$\mathscr{H}_L$	Hankel matrix with L block-rows
σ	backward shift $(\sigma w)(t) := w(t+1)$
$\Sigma_{m,n}(w_{\rm d})$	$\{\mathscr{B} : w_{\rm d} \in \mathscr{B}\vert_T, \mathscr{B} \in \mathscr{L}_m^{q,n}\}$
$\mathrm{MPUM}(w_{\rm d})$	$\arg\min_{\mathscr{B} \in \mathscr{L}^q, w_{\rm d} \in \mathscr{B}} \mathbf{c}(\mathscr{B})$
$\mathrm{MPUM}_m(w_{\rm d})$	$\arg\min_{\mathscr{B} \in \mathscr{L}_m^q, w_{\rm d} \in \mathscr{B}\vert_T} \mathbf{n}(\mathscr{B})$

III. Most powerful unfalsified model

The original definition [1, Definition 4] of the MPUM is for an infinite time series $w_{\rm d} \in (\mathbb{R}^q)^{\mathbb{N}}$ in the model class $\mathscr{L}^q$:

$$\mathrm{MPUM}(w_{\rm d}) := \arg\min_{\mathscr{B} \in \mathscr{L}^q, w_{\rm d} \in \mathscr{B}} \mathbf{c}(\mathscr{B}). \tag{1}$$

In this case, $\mathrm{MPUM}(w_{\rm d})$ exists and is unique. It is given by the span of $w_{\rm d}$ and all its shifts:

$$\mathrm{MPUM}(w_{\rm d}) = \mathrm{span}\{ w_{\rm d}, \sigma w_{\rm d}, \dots, \sigma^t w_{\rm d}, \dots \}.$$

In case of a finite data $w_{\rm d} \in (\mathbb{R}^q)^T$, however, $\mathrm{MPUM}(w_{\rm d})$ is always an autonomous system. The reason for this is that in the definition of LTI system's complexity $\mathbf{c}(\mathscr{B})$ the ordering is lexicographic (number of inputs $\mathbf{m}(\mathscr{B})$ has precedence over the order $\mathbf{n}(\mathscr{B})$), however, every finite time series can be fitted exactly by a finite dimensional autonomous LTI system. Therefore, there are exact autonomous models $\mathscr{B} \in \mathscr{L}^q$ that are by definition less complex than any open model.

One approach to resolve this issue is to assume that the number of inputs is a priori known and define the MPUM as minimization of the order:

$$\mathrm{MPUM}_m(w_{\rm d}) := \arg\min_{\mathscr{B} \in \mathscr{L}_m^q, w_{\rm d} \in \mathscr{B}|_T} \mathbf{n}(\mathscr{B}). \tag{2}$$

Lemma 1. *For any finite time series $w_d \in (\mathbb{R}^q)^T$, $\mathrm{MPUM}_m(w_d)$ exists and is unique.*

Proof. Existence follows from the facts that $w_{\rm d}$ is finite and the model class $\mathscr{L}_m^q$ allows arbitrary high model order. Uniqueness follows from the facts that $\mathrm{MPUM}_m(w_{\rm d})$ is equal to the intersection of all exact models for $w_{\rm d}$ in $\mathscr{L}_m^q$ and $\mathscr{L}_m^q$ is closed. □

Remark 1 (Detecting the number of inputs from data)**.** *The number of inputs can be found from data by computing the rank of Hankel matrices $\mathscr{H}_L(w_d)$ for different values of L. Indeed, under the identifiability conditions of [2], for sufficiently large L (larger than the lag of the system), we have that*

$$\mathrm{rank}\, \mathscr{H}_L(w_d) = \mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B}).$$

Therefore, $\mathbf{m}(\mathscr{B})$ and $\mathbf{n}(\mathscr{B})$ can be computed from two values of the rank of the Hankel matrix: $\mathrm{rank}\, \mathscr{H}_{L_1}(w_d)$ and $\mathrm{rank}\, \mathscr{H}_{L_2}(w_d)$ with $L_1 \neq L_2$ and $L_1, L_2 > \mathbf{l}(\mathscr{B})$.

An exact model in a model class of bounded complexity may not exist or be nonunique (see Section V). We define the set of exact models of bounded complexity as

$$\Sigma_{m,n}(w_{\rm d}) = \{\mathscr{B} : w_{\rm d} \in \mathscr{B}|_T, \mathscr{B} \in \mathscr{L}_m^{q,n}\}. \tag{3}$$

Our goal is to characterize the set $\Sigma_{m,n}(w_{\rm d})$. We prove (see Theorem 5) that

$$\Sigma_{m,n}(w_{\rm d}) = \mathrm{MPUM}_m(w_{\rm d}) + \mathscr{L}_0^{q,n-k}, \quad \text{where } k = \mathbf{n}\big(\mathrm{MPUM}_m(w_{\rm d})\big). \tag{4}$$

That is, the set of models explaining the data $w_{\rm d}$ is the sum of the MPUM and the set of autonomous models of order $n - \mathbf{n}\big(\mathrm{MPUM}_m(w_{\rm d})\big)$.

IV. Autonomous behaviors

A. The Scalar Case

First, we consider the scalar case $q = 1$.

Lemma 2. *Let $w_d \in \mathbb{R}^T$ be a given time series that is generated by $\mathscr{B} \in \mathscr{L}_0^{1,n}$. Then the order of $\mathrm{MPUM}_0(w_d)$ is equal to the persistency of excitation order of w_d.*

Proof. Let k be the persistency of excitation order of w_d, *i.e.*, $\operatorname{rank} \mathscr{H}_k(w_\mathrm{d}) = k$ and $\operatorname{rank} \mathscr{H}_{k+1}(w_\mathrm{d}) = k$. We need to show that $\mathbf{n}\big(\mathrm{MPUM}_0(w_\mathrm{d})\big) = k$. By the rank deficiency of $\mathscr{H}_{k+1}(w_\mathrm{d})$, there exists $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_k \end{bmatrix} \neq 0$, such that

$$P \mathscr{H}_{k+1}(w_\mathrm{d}) = 0. \quad (5)$$

Note that $P_k \neq 0$ otherwise $\operatorname{rank} \mathscr{H}_k(w_\mathrm{d}) < k$. From (5), define a behavior $\mathscr{B}$ whose kernel representation is given by $P(\xi) = \Sigma_{i=0}^k P_i \xi^i$. Clearly, $\mathbf{n}(\mathscr{B}) = \deg P(\xi) = k$. Since $w_\mathrm{d} \in \mathscr{B}|_T$ and $\mathscr{B} \in \mathscr{L}_0^{1,n}$, $\mathscr{B}$ is an exact model for the data.

Next, we need to prove that the behavior $\mathscr{B}$ is the most powerful, *i.e.*, there is no other exact model of lower order than k. This follows from the fact that if there exists a model of order less than k, then $\operatorname{rank} \mathscr{H}_k(w_\mathrm{d}) < k$, which is a contradiction. Hence, $\mathscr{B}$ is the MPUM. □

The following theorem characterizes the set of all exact models in the scalar case.

Theorem 1. *The set of exact autonomous models of complexity bounded by n for a scalar time series w_d is given as*

$$\Sigma_{0,n}(w_d) = \mathrm{MPUM}_0(w_d) + \mathscr{L}_0^{1,n-k}, \quad \text{where } k = \mathbf{n}\big(\mathrm{MPUM}_0(w_d)\big). \quad (6)$$

Proof. The proof is similar to the one of Theorem 3 and is skipped. □

Theorem 1 indicates that every exact model must include the MPUM. In the special case $k = n$, the model class $\mathscr{L}_0^{q,0}$ contains only the trivial behavior $\mathscr{B} = \{0\}$, and thus

$$\Sigma_{0,n}(w_\mathrm{d}) = \mathrm{MPUM}_0(w_\mathrm{d}) + \{0\} = \mathrm{MPUM}_0(w_\mathrm{d}).$$

Corollary 1. *There is a unique exact model $\Sigma_{0,n}(w_d) = \mathrm{MPUM}_0(w_d)$ if and only if w_d is persistently exciting of order n.*

B. The Multivariable Case

It turns out that Lemma 2 does not hold true in the case of multivariable $(q > 1)$ autonomous systems. However, we have the following theorem for the multivariable case.

Theorem 2. *Let $w_d \in (\mathbb{R}^q)^T$ be a time series that is generated by $\mathscr{B} \in \mathscr{L}_0^{q,n}$,* i.e.*, $w_d \in \mathscr{B}|_T$. Then,*

$$\mathrm{MPUM}_0(w_d) = \mathscr{B} \quad \text{and} \quad \operatorname{span} \mathscr{H}_L(w_d) = \mathscr{B}|_L, \quad \text{for } L \leq L^*$$

if and only if $\operatorname{rank} \mathscr{H}_{L^*}(w_d) = \mathbf{n}(\mathscr{B})$. Further,

$$\mathbf{n}\big(\mathrm{MPUM}_0(w_d)\big) = \operatorname{rank} \mathscr{H}_{L^*}(w_d). \quad (7)$$

Proof. Clearly, $\operatorname{span} \mathscr{H}_{L^*}(w_\mathrm{d}) \subseteq \mathscr{B}|_{L^*}$. By the rank condition $\operatorname{rank} \mathscr{H}_{L^*}(w_\mathrm{d}) = \mathbf{n}(\mathscr{B})$, it follows that $L^* \geq \mathbf{l}(\mathscr{B})$ and

$$\dim\big(\operatorname{span} \mathscr{H}_{L^*}(w_\mathrm{d})\big) = \operatorname{rank} \mathscr{H}_{L^*}(w_\mathrm{d}) = \mathbf{n}(\mathscr{B}) = \dim(\mathscr{B}|_{L^*}).$$

Hence, $\operatorname{span} \mathscr{H}_{L^*}(w_\mathrm{d}) = \mathscr{B}|_{L^*}$ and therefore $\operatorname{span} \mathscr{H}_L(w_\mathrm{d}) = \mathscr{B}|_L$, for all $L \leq L^*$.

Next, we prove that $\mathrm{MPUM}_0(w_\mathrm{d}) = \mathscr{B}$. Since $\mathscr{B} \in \mathscr{L}_0^{q,n}$ and $w_\mathrm{d} \in \mathscr{B}|_T$, it remains to show that $\mathscr{B}$ is most powerful. Let there be another exact model $\bar{\mathscr{B}} \in \mathscr{L}_0^{q,n}$, such that $\mathbf{n}(\bar{\mathscr{B}}) < \mathbf{n}(\mathscr{B})$. Then, $\operatorname{rank} \mathscr{H}_{L^*}(w_\mathrm{d}) < \mathbf{n}(\mathscr{B})$, which is a contradiction. Hence, $\mathrm{MPUM}_0(w_\mathrm{d}) = \mathscr{B}$.

Conversely, $\mathrm{MPUM}_0(w_\mathrm{d}) = \mathscr{B}$ implies $w_\mathrm{d} \in \mathscr{B}|_T$. Next, $\operatorname{span} \mathscr{H}_{L^*}(w_\mathrm{d}) = \mathscr{B}|_{L^*}$ implies $\operatorname{rank} \mathscr{H}_{L^*}(w_\mathrm{d}) = \dim(\mathscr{B}|_{L^*}) = \mathbf{n}(\mathscr{B})$. Therefore, (7) holds. □

Next, we state and prove a generalization of Theorem 1 to multivariable autonomous systems.

Theorem 3. *The set of exact autonomous models of complexity bounded by n for a time series $w_d \in (\mathbb{R}^q)^T$ is given as*

$$\Sigma_{0,n}(w_d) = \mathrm{MPUM}_0(w_d) + \mathscr{L}_0^{q,n-k}, \quad \text{where } k = \mathbf{n}\big(\mathrm{MPUM}_0(w_d)\big). \quad (8)$$

Proof. Clearly,

$$\mathrm{MPUM}_0(w_\mathrm{d}) + \mathscr{L}_0^{q,n-k} \subseteq \Sigma_{0,n}(w_\mathrm{d}).$$

Next, we prove the reverse inclusion. Let $\mathscr{B} \in \Sigma_{0,n}(w_\mathrm{d})$ and

$$\mathbf{n}\big(\mathrm{MPUM}_0(w_\mathrm{d})\big) = k \leq \mathbf{n}(\mathscr{B}) \leq n.$$

Then, $\mathrm{MPUM}_0(w_\mathrm{d}) \subseteq \mathscr{B}$. Thus,

$$\mathscr{B} = \mathrm{MPUM}_0(w_\mathrm{d}) + \mathscr{B}', \quad \text{where } \mathscr{B}' \in \mathscr{L}_0^{q,n-k},$$

and hence (8) holds. □

V. Open behaviors

Analogous to Theorem 2, a theorem for open systems is stated below.

Theorem 4. *Let $w_d \in (\mathbb{R}^q)^T$ be a given time series that is generated by a linear time-invariant system $\mathscr{B} \in \mathscr{L}_m^{q,n}$ with $\mathbf{m}(\mathscr{B}) = m$ inputs,* i.e.*, $w_d \in \mathscr{B}|_T$. Then*

$$\mathrm{MPUM}_m(w_d) = \mathscr{B} \quad \text{and} \quad \operatorname{span} \mathscr{H}_L(w_d) = \mathscr{B}|_L, \quad \text{for } L \leq L^*$$

if and only if $\operatorname{rank} \mathscr{H}_{L^*}(w_d) = \mathbf{n}(\mathscr{B}) + mL^*$. Further,

$$\mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big) = \operatorname{rank} \mathscr{H}_{L^*}(w_d) - mL^*. \quad (9)$$

Proof. The proof is similar to the one of Theorem 2, taking into account that $\dim(\mathscr{B}|_L) = \mathbf{n}(\mathscr{B}) + mL$, for $L \geq \mathbf{l}(\mathscr{B})$. □

Analogous to Theorem 3, we have the following theorem that provides a characterization of the set of exact models in case of open systems.

Theorem 5. *The set of exact open models of complexity bounded by n for a given time series $w_d \in (\mathbb{R}^q)^T$ is given as*

$$\Sigma_{m,n}(w_d) = \mathrm{MPUM}_m(w_d) + \mathscr{L}_0^{q,n-k}, \quad \text{where } k = \mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big). \quad (10)$$

We can distinguish three cases:

1) if $n < \mathbf{n}\big(\mathrm{MPUM}_m(w_\mathrm{d})\big)$, there is no exact model, *i.e.*, $\Sigma_{m,n}(w_\mathrm{d}) = \emptyset$,

2) if $n = \mathbf{n}\big(\mathrm{MPUM}_m(w_\mathrm{d})\big)$, there is a unique exact model, *i.e.*, $\Sigma_{m,n}(w_\mathrm{d}) = \mathrm{MPUM}_m(w_\mathrm{d})$, and
3) if $n > \mathbf{n}\big(\mathrm{MPUM}_m(w_\mathrm{d})\big)$, there are infinitely many exact models (10).

Note that, in case of nonunique exact model (10), except possibly for the MPUM, all exact models are uncontrollable.

VI. Examples

Example 1 (Scalar autonomous behavior)**.** *Let $w_d \in \mathbb{R}^T$ be a trajectory of a k-th order scalar autonomous LTI system $\mathscr{B}$. Assuming that w_d is persistently exciting of order k, by Lemma 2, the MPUM of w_d coincides with the data generating system $\mathscr{B}$. Also, the Hankel matrix $\mathscr{H}_{k+1}(w_d)$ has one dimensional left kernel. Let $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_k \end{bmatrix}$, be a nonzero vector, such that $P\mathscr{H}_{k+1}(w_d) = 0$. The polynomial $P(\xi) = \Sigma_{i=0}^{k} P_i \xi^i$, defined by the vector P, yields a kernel representation of* $\mathrm{MPUM}_0(w_d)$.

Since we are interested in exact models of order n, we consider the extended Hankel matrix $\mathscr{H}_{n+1}(w_d)$ with $n+1$ rows. The set of nonzero vectors in the left kernel of $\mathscr{H}_{n+1}(w_d)$ identify the set of exact scalar autonomous LTI models for w_d of order at most n. Define the $(n-k+1)\times(n+1)$ generalized Sylvester matrix of P

$$\mathscr{S}(P) := \begin{bmatrix} P_0 & P_1 & \cdots & P_k & & & \\ & P_0 & P_1 & \cdots & P_k & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & P_0 & P_1 & \cdots & P_k \end{bmatrix}.$$

By the definition of P and the Hankel structure, we have that

$$\mathscr{S}(P)\mathscr{H}_{n+1}(w_d) = 0.$$

Moreover, since the left kernel of $\mathscr{H}_{n+1}(w_d)$ has dimension $n-k+1$, the rows of $\mathscr{S}(P)$ define a basis. Therefore, any $\hat{P}$ such that $\hat{P}\mathscr{H}_{n+1}(w_d) = 0$, can be written as

$$\hat{P} = z\mathscr{S}(P), \quad \textit{for some } z \in \mathbb{R}^{n-k+1}, \tag{11}$$

i.e., *any exact model for w_d in the model class $\mathscr{L}_0^{1,n}$ has a kernel representation with the structure (11).*

The structure (11) of the left kernel implies that the polynomial $\hat{P}(\xi) = \sum_{i=0}^{n} \hat{P}_i \xi^i$ defined by the vector $\hat{P}$ has k "fixed" roots coinciding with the roots of $P(\xi)$ and $n-k$ "spurious" roots determined by z. The fixed roots are the roots of the MPUM while the spurious roots correspond to the roots of an $n-k$-th order autonomous system. This result about the left kernel of the Hankel matrix of the data corresponds to Theorem 1 (see (6)*).*

Example 2 (SISO system)**.** *Consider a controllable single-input single-output system $\mathscr{B} \in \mathscr{L}_1^{2,k}$ defined by the input/output representation: $P(\sigma)y = Q(\sigma)u$, where P, Q, with* $\det P \neq 0$*, are scalar polynomials. Equivalently,*

$$\underbrace{\begin{bmatrix} -Q(\sigma) & P(\sigma) \end{bmatrix}}_{R(\sigma)} \begin{bmatrix} u \\ y \end{bmatrix} = 0,$$

is a kernel representation of $\mathscr{B}$. Let $w_d := \left[\begin{smallmatrix} u_d \\ y_d \end{smallmatrix}\right] \in (\mathbb{R}^2)^T$ be a trajectory of $\mathscr{B}$ that satisfies the identifiability conditions of [2]. Then, we have that $\mathscr{H}_{k+1}(w_d)$ has one-dimensional left kernel and $R\mathscr{H}_{k+1}(w_d) = 0$.

As in the scalar case, the Hankel matrix with $n+1$ block-rows $\mathscr{H}_{n+1}(w_d)$ has $n-k+1$ dimensional left kernel. A basis for the left kernel of $\mathscr{H}_{n+1}(w_d)$ is given by the rows of the generalized Sylvester matrix $\mathscr{S}(R)$. Any $\hat{R}$ in the left kernel of $\mathscr{H}_{n+1}(w_d)$ is given by $\hat{R} = z\mathscr{S}(R)$, where $z \in \mathbb{R}^{n-k+1}$. Since $\hat{R}$ correspond to an exact model for w_d in $\mathscr{L}_1^{2,n}$, we have again special structure of the set of exact models due to the special structure of the left kernel of $\mathscr{H}_{n+1}(w_d)$. Note that, with $\hat{R} =: \begin{bmatrix} -\hat{Q} & \hat{P} \end{bmatrix}$, $\hat{P}$ and $\hat{Q}$ are such that $\hat{P} = zP$ and $\hat{Q} = zQ$. This corresponds to Theorem 5 (see (10)*): the k "fixed" poles and zeros are the ones of the MPUM and the $n-k$ "spurious" ones are common for the $\hat{P}$ and $\hat{Q}$. In particular, this shows that, except for the MPUM, the exact models for w_d in $\mathscr{L}_1^{2,n}$ are uncontrollable.*

VII. Conclusions

We have investigated the problem of exact system identification for a given finite time series. It has been proven that any exact model must include the MPUM. We have modified the notion of the MPUM for a finite time series assuming that the number of inputs is given and proved a new identifiability result with the modified MPUM. Our approach is algebraic and utilizes concepts from the behavioral systems theory. Most of the results have been related to computing the rank of the Hankel matrix constructed from the given time series. It is notable that the work does not require a partition of the input/output variables. Examples are given to illustrate the developed results.

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