A comparison between structured low-rank approximation and correlation approach for data-driven output tracking

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Abstract: Data-driven control is an alternative to the classical model-based control paradigm. The main idea is that a model of the plant is not explicitly identified prior to designing the control signal. Two recently proposed methods for data-driven control—a method based on correlation analysis and a method based on structured matrix low-rank approximation and completion—solve identical control problems. The aim of this paper is to compare the methods, both theoretically and via a numerical case study. The main conclusion of the comparison is that there is no universally best method: the two approaches have complementary advantages and disadvantages. Future work will aim to combine the two methods into a more effective unified approach for data-driven output tracking.

Keywords: data-driven control, output tracking, virtual reference feedback tuning, structured low-rank approximation, matrix completion.

1. INTRODUCTION

In the last 25 years, the system identification community has been searching for rigorous solutions to the issue of the interplay of identification and model-based design by asking the question: *what is the best model for control?* This research direction is aiming to fix the shortcoming of the model-based approach that the model is not tailored for its intended use.

Data-driven methods take a different approach: *they solve the design problem without splitting it into identification and model-based design.* The issue of developing identification methods aimed at their intended usage is considered by the system identification community in an area of research known as *identification for control*, see Ljung (2002); Gevers (2005). The identified model is tuned for optimal performance of the closed-loop system, *i.e.*, the identification criterion is linked with the control objective. The interplay between identification and control is central also in *dual adaptive control* (Åström and Wittenmark (2008)), where the modeling and control tasks are solved simultaneously, in real-time. Both identification for control and adaptive control, however, are *model-based methods:* they aim at a model as an intermediate step towards control.

An alternative to the model-based approach is to design a controller directly from data without first identifying a model. This approach, known as model-free or *direct data-driven control*, has its roots in classical heuristics for PID controller tuning such as the Ziegler-Nichols method in Ziegler and Nichols (1942). Rigorous data-driven control methods, however, appeared only since the late 90's. Traditionally, they fall into the following three main approaches.

(1) *Adaptive control* like Direct Model Reference Adaptive Control (Direct MRAC, see Landau et al. (1998)) and Unfalsified control (see Safonov and Tsao (1997) and related works). The former is a reformulation of classical MRAC, where the controller parameters are directly taken into account. The latter is an adaptive approach, where the controller is viewed as an exclusion rule (see Willems (1986, 1987)) and the main idea is to reject (falsify) controllers using previously collected experimental data from the plant.

(2) *Off-line iterative design*, like Iterative Feedback Tuning in Hjalmarsson et al. (1998) or Iterative Correlation-based Tuning in Karimi et al. (2004), optimizes the controller parameters by a gradient type algorithm that uses the control objective. The key feature of this approach is that the control objective and its gradient are evaluated using only measured data, obtained from special experiments.

(3) *Off-line noniterative design*, like Virtual Reference Feedback Tuning (VRFT, see Campi et al. (2002)) or noniterative unfalsified control (see Battistelli et al. (2017)), are *one-shot* approaches that optimize the parameters of a fixed-structure controller directly from a set of input/output data. Recently, an approach based on missing data estimation and a correlation-based method were proposed (see, respectively, Markovsky (2017b) and Van Heusden et al. (2011)).

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The aim of this paper is to make a comparison between the missing data estimation and correlation approaches. The former problem is posed and solved in the structured low-rank approximation setting of Markovsky (2008). The resulting methods are implemented in Markovsky and Usevich (2014).

The paper is organized as follows. The control problem is formally stated in Section 2. Section 3 briefly recalls the approaches of Markovsky (2017b) and Van Heusden et al. (2011). A theoretical and numerical comparison are given in Section 4 and Section 5, respectively. The paper is ended by some concluding remarks.

2. PROBLEM STATEMENT

We use the behavioral language of Willems (1986, 1987). A discrete-time dynamical system $\mathscr{B}$ with q external variables (inputs and outputs) is a subset of the signal space $(\mathbb{R}^q)^{\mathbb{N}}$. The set of natural numbers $\mathbb{N}$ is the time-axis, so that a trajectory w of $\mathscr{B}$ is a vector time series $w = \big(w(1), w(2), \dots\big)$, where $w(t) \in \mathbb{R}^q$, for all $t \in \mathbb{N}$. The notation $\mathscr{B}_t$ is used for the restriction of the behavior on the interval $[1,t]$, *i.e.*,

$$\mathscr{B}_t := \{\, w_{\rm p} \in (\mathbb{R}^q)^t \mid \text{ there is } w_{\rm f}, \text{ such that } w_{\rm p} \wedge w_{\rm f} \in \mathscr{B} \,\},$$

where $w_{\rm p} \wedge w_{\rm f}$ denotes the concatenation of the trajectories $w_{\rm p}$ and $w_{\rm f}$. Thus, $w \in \mathscr{B}_t$ is a finite trajectory $\big(w(1), \dots w(t)\big)$ of the system $\mathscr{B}$.

The number of inputs $\mathtt{m}$ and the number of outputs $\mathtt{p}$ of a system $\mathscr{B} \in (\mathbb{R}^{\mathtt{m}+\mathtt{p}})^{\mathbb{N}}$ are invariant of the representation. After permutation of the variables, a trajectory w of the system $\mathscr{B}$ can be partition as

$$w = (u, y) = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right],$$

where u is an input, *i.e.*, it is free, and y is an output, *i.e.*, it is determined by the input, the system, and the initial condition.

Problem 1 (Model based output tracking). *Given a linear time-invariant system $\mathscr{B}$, initial conditions w_p and an output y_f generated according to a desired dynamics $\mathscr{B}_M$, find a control input signal $\widehat{u}_f$ that*

$$\begin{array}{ll} \textit{minimize} & \|y_f - \widehat{y}_f\| \\ \textit{subject to} & w_{ini} \wedge (\widehat{u}_f, \widehat{y}_f) \in \mathscr{B}, \end{array} \qquad \text{(CTR)}$$

i.e., *applying $\widehat{u}_f$ on the system $\mathscr{B}$ under initial conditions w_p, the resulting output $\widehat{y}_f$ is closest to y_f in the 2-norm sense.*

Note that the signal $u_{\rm f}$ is the *open-loop* optimal control signal. In practice, such a signal can be used in a model predictive control setting, which will implicitly create a closed loop and make the overall control scheme robust to disturbances, measurement noise and uncertainty on model or initial conditions.

We denote by $\mathscr{L}^q$ the class of linear time-invariant finite dimensional systems with q variables. $\mathscr{L}^q_{\mathtt{m},\ell}$ denotes the subclass of $\mathscr{L}^q$ with bounded complexity: at most $\mathtt{m}$ inputs and lag (observability index) at most ℓ. The most powerful unfalsified model, defined in Willems (1986), is denoted by $\mathscr{B}_{\rm mpum}(w_{\rm d})$. $\mathscr{B}_{\rm mpum}(w_{\rm d})$ is the linear time-invariant system $\mathscr{B}$ of minimal complexity that is exact for the data, *i.e.*, $w_{\rm d} \in \mathscr{B}$.

Problem 2 (Data-driven output tracking). *Given a complexity specification $(\mathtt{m}, \ell)$, a trajectory w_d of an unknown system $\mathscr{B} \in \mathscr{L}_{\mathtt{m},\ell}$, initial conditions w_p and an output y_f generated according to a desired dynamics $\mathscr{B}_M$, find a signal $\widehat{u}_f$ that*

$$\begin{array}{ll} \textit{minimize} & \|y_f - \widehat{y}_f\| \\ \textit{subject to} & w_{ini} \wedge (\widehat{u}_f, \widehat{y}_f) \in \mathscr{B}_{mpum}(w_d) \in \mathscr{L}_{\mathtt{m},\ell}. \end{array} \qquad \text{(DD CTR)}$$

In the data-driven control problem formulation, we assume that the trajectory $w_{\rm d}$ is exact and satisfies the identifiability conditions of Willems et al. (2005). In case of noisy data, the problem should include also approximation of $w_{\rm d}$, *e.g.*, if the data is generated in the errors-in-variables setting with white Gaussian noise of equal variance on all variables, the maximum likelihood estimator is defined by

$$\begin{array}{ll} \text{minimize} & \|w_{\rm d} - \widehat{w}_{\rm d}\|_2^2 + \|y_{\rm f} - \widehat{y}_{\rm f}\|_2^2 \\ \text{subject to} & w_{\rm ini} \wedge (\widehat{u}_{\rm f}, \widehat{y}_{\rm f}) \in \mathscr{B}_{\rm mpum}(w_{\rm d}) \in \mathscr{L}_{\mathtt{m},\ell}. \end{array} \qquad \text{(DD CTR')}$$

3. THE METHODS

3.1 Method based on structured low-rank approximation

In Markovsky (2017b), it is shown that the data-driven control problem can be posed and solved as a structured weighted low-rank matrix approximation problem. The key result, presented first, is a link between a trajectory of a linear time-invariant system and rank deficiency of a mosaic-Hankel matrix. Then, we present a method based on the variable projections principle in Golub and Pereyra (2003).

Link to element-wise weighted mosaic-Hankel structured low-rank approximation. In order to solve (DD CTR), we use the equivalence of trajectories of a linear time-invariant system with bounded complexity and rank deficiency of a matrix constructed from the trajectories.

Lemma 3 (Markovsky (2013)). *Let $\mathtt{m}$, $\mathtt{p}$, and ℓ be, respectively, the number of inputs, the number of outputs, and the lag of a linear time-invariant system $\mathscr{B}$. Then,*

$$w^1, w^2 \in \mathscr{B} \iff \textit{rank}\big(\, [\mathscr{H}(w^1) \;\; \mathscr{H}(w^2)] \,\big) \leq (\mathtt{m}+\mathtt{p})\ell + \mathtt{m},$$

where $\mathscr{H}(\cdot)$ is a block-Hankel matrix

$$\mathscr{H}(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-\ell) \\ w(2) & w(3) & \cdots & w(T-\ell+1) \\ w(3) & w(4) & \cdots & w(T-\ell+2) \\ \vdots & \vdots & & \vdots \\ w(\ell+1) & w(\ell+2) & \cdots & w(T) \end{bmatrix}.$$

Using Lemma 3, it can be shown that the data-driven control problem (DD CTR') is equivalent to the structured weighted low-rank approximation problem

$$\begin{array}{ll} \text{minimize} & \text{over } \widehat{p} \quad \|p - \widehat{p}\|_v^2 \\ \text{subject to} & \text{rank}\big(\mathscr{S}(\widehat{p})\big) \leq r, \end{array} \qquad \text{(WSLRA)}$$

with

- data vector $p = (w_{\rm d}, w)$,
- structure $\mathscr{S}(\widehat{p}) = [\mathscr{H}(\widehat{w}_{\rm d}) \;\; \mathscr{H}(\widehat{w})]$,
- rank constraint $r = (\mathtt{m}+\mathtt{p})\ell + \mathtt{m}$, and
- element-wise weighted semi-norm

$$\|p - \widehat{p}\|_v := \|w_{\rm d} - \widehat{w}_{\rm d}\|_2^2 + \|y_{\rm f} - \widehat{y}_{\rm f}\|_2^2.$$

Problem (WSLRA) is a nonconvex optimization problem. It can be solved by convex relaxation, using the nuclear norm heuristic, subspace methods and local optimization methods. Next, we describe a local optimization method, based on the variable projections principle.

Solution method based on the variable projections. Let $\mathscr{S}(p)$ be an $m \times n$ matrix. First, we express the rank constraint in (WSLRA) as a condition on the dimension of the left kernel

$$\operatorname{rank}\big(\mathscr{S}(\widehat{p})\big) \leq r \quad \iff \quad \exists \text{ full row rank } R \in \mathbb{R}^{(m-r)\times m}, \text{ such that } R\mathscr{S}(\widehat{p}) = 0. \tag{rank$_R$}$$

Then, using (rank$_R$), we rewrite (WSLRA) in the following equivalent form

$$\begin{aligned} &\text{minimize over } \widehat{p}, R \in \mathbb{R}^{(m-r)\times m} \ \|p-\widehat{p}\|_v \\ &\text{subject to} \quad R\mathscr{S}(\widehat{p}) = 0 \\ &\text{and} \quad R \text{ is full row rank.} \end{aligned} \tag{WSLRA$_R$}$$

The variable $\widehat{p}$ can be eliminated by representing (WSLRA$_R$) as a double minimization problem:

$$\text{minimize} \quad \text{over full row rank } R \in \mathbb{R}^{(m-r)\times m} \quad M(R), \tag{OUTER}$$

where

$$M(R) := \min_{\widehat{p}} \quad \|p-\widehat{p}\|_2^2 \quad \text{subject to} \quad R\mathscr{S}(\widehat{p}) = 0. \tag{INNER}$$

Solution of (INNER), *i.e.*, evaluation of $M(R)$ for given R, is refered to as the *inner minimization*. Solution of (OUTER), *i.e.*, optimization of M over R, is referred to as the *outer minimization*.

The inner minimization (INNER) is a generalized linear least squares problem (see Paige (1979a,b); Lawson and Hanson (1987)) and admits an analytic solution, as indicated in Markovsky and Usevich (2013).

The main advantage in the reformulation of (WSLRA$_R$) as (OUTER) is the elimination of the optimization variable $\widehat{p}$. In control applications, $\widehat{w}_\text{d}$ and $\widehat{w}$ are high dimensional and R is small dimensional. Therefore, the elimination of $\widehat{w}_\text{d}$ and $\widehat{w}$ leads to a big reduction in the number of the optimization variables. The approach described above for solving (WSLRA$_R$) is similar to the variable projection method of Golub and Pereyra (2003) for the solution of separable unconstrained non-linear least squares problems.

In (OUTER), the cost function M is minimized over the set of full row rank matrices R. (INNER) of $M(R)$ depends only on the space spanned by the rows of R, *i.e.*, $M(R) = M(UR)$, for all nonsingular $U \in \mathbb{R}^{(m-r)\times(m-r)}$. Therefore, (OUTER) is a minimization problem on the Grassmann manifold of all r-dimensional subspaces of $\mathbb{R}^m$. In order to find a minimum of M, the search space in (OUTER) can be replaced by a set of matrices $R \in \mathbb{R}^{p\times q(\ell+1)}$ that represent all subspaces of the Grassmann manifold, *e.g.*, all matrices satisfying the constraint $RR^\top = I_{m-r}$.

A software package for solving weighted mosaic-Hankel structured low-rank approximation problems (WSLRA), based on the variable projections approach, is developed in Markovsky and Usevich (2014). This package is used for solution of linear time-invariant system identification problems (see Markovsky (2013)). The main functions of this latter package are `ident`, which solves problem (DD CTR') and `misfit`, which solves the inner minimization problem (INNER). The software is available online at `http://slra.github.io/`

3.2 Noniterative correlation-based tuning method

The objective of noniterative correlation-based tuning is to design a suitable control signal $\widehat{u}_\text{f}$ as a feedback signal. Therefore, the optimization variable here is not the signal but the controller producing $\widehat{u}_\text{f}$ when fed by the mismatch between the reference signal and the measured output. More specifically, a linear, fixed-order controller $K(q^{-1},\rho)$, linearly parameterized through $\rho \in \mathbb{R}^n$, is considered. Formally, the controller parameterization is $K(q^{-1},\rho) = \beta^T(q^{-1})\rho$, where $\beta(q^{-1})$ is a vector of n linear discrete-time transfer operators and q^{-1} is the backward shift operator. The key assumption is again that an open-loop collection of input-output (I/O) data w_d, with output affected by additive stationary noise, is available.

The Correlation-based Tuning (CbT) rationale is as follows. Let $G(q^{-1})$ and $M(q^{-1})$ be the transfer operators of $\mathscr{B}$ and $\mathscr{B}_M$, respectively, and consider the closed-loop model matching error in the 2-norm sense

$$J_{mm}(\rho) = \left\| M - \frac{GK(\rho)}{1+GK(\rho)} \right\|, \tag{CLMM}$$

as a function of ρ. Under the assumption that $\mathscr{B}_M$ is achievable, it can be shown that the minimizer of (CLMM) is the same of $J(\rho) = \|\Delta(\rho)\|$, where

$$\Delta = M - (1-M)GK(\rho), \tag{DELTA}$$

since, in the minimum, the sensitivity function coincides with the ideal one, *i.e.*, $1-M$.

The most important observation at the basis of the CbT approach is that, in the noiseless setting, the closed-loop model matching error $\varepsilon(t,\rho)$ can be directly minimized from data. In fact, consider the output $\varepsilon(t,\rho)$ of $\Delta(\rho)$ fed by $r(t)$:

$$\varepsilon(t,\rho) = Mr(t) - (1-M)K(\rho)Gr(t).$$

It can be shown that such an output can be computed without the need of the knowledge of G, in case the available data are used $r(t) = u_\text{f}(t)$. In fact, the error $\varepsilon_\text{f}(t,\rho)$ corresponding to the use of w_d reads

$$\varepsilon_\text{f}(t,\rho) = Mu_\text{f}(t) - (1-M))K(\rho)y_\text{f}(t),$$

since y_f is the output of G when fed by u_f. Then, the minimizer of the $\mathscr{L}_2$-norm of $\varepsilon_\text{f}(t,\rho)$ corresponds exactly to $K_o(q^{-1})$, provided that u_f has a flat spectrum.

When data are collected in a noisy environment, the method resorts to the correlation approach to identify the controller. Specifically, an extended instrumental variable $\zeta(t)$ correlated with $u_\text{f}(t)$ and uncorrelated with the output disturbance is introduced to decorrelate the error signal $\varepsilon(t)$ and $u_\text{f}(t)$. $\zeta(t)$ is defined as $\zeta(t) = [u_\text{f}(t+l),\ldots,u_\text{f}(t),\ldots,u_\text{f}(t-l)]^T$, where l is a sufficiently large integer. The correlation function is defined as

$$f_{N,l}(\rho) = \frac{1}{N}\sum_{t=1}^{N} \zeta(t)\varepsilon(t,\rho) \tag{CF}$$

and the correlation criterion as

$$J_{N,l}(\rho) = f_{N,l}^T(\rho) f_{N,l}(\rho). \tag{CC}$$

In Van Heusden et al. (2011), it has been proven that

$$\lim_{N,l\to\infty,\, l/N\to 0} J_{N,l}(\rho) = J_{mm}(\rho),$$

for any input sequence, if the reference behaviour is achievable and the data in $\zeta(t)$ are prefiltered by $L(q^{-1})$, defined as

$$L(e^{-j\omega}) = \frac{1-M(e^{-j\omega})}{\Phi_u(\omega)}, \tag{L}$$

where $\Phi_u(\omega)$ denotes the spectral density of $u_\text{f}(t)$. Notice that such a prefilter may be non-causal but it can be implemented off-line. For further details about the performance of CbT, see also Formentin et al. (2013); Formentin and Karimi (2014).

4. A THEORETICAL COMPARISON

Both the SLRA and CbT methods, presented in the previous section, aim at solving the output tracking Problem 2; however, they use rather different approaches. Also, the literature where they have been developed is disjoint. Because of this, it is not clear how the methods compare and in particular what their advantages and disadvantages are. This section compares the two approaches. An overview of the comparison is shown in Table 1.

The CbT method was originally conceived for single-input single-output linear time-invariant systems. Since the derivation of the CbT formulas is based on the commutation of controller and plant blocks, CbT cannot be straightforwardly extended to multivariable or nonlinear plants. The SLRA method is instead applicable to multi-input multi-output systems linear time-invariant systems. There are also preliminary results on the generalization of the SLRA approach to the class of nonlinear systems defined by a polynomial difference equation, see Markovsky (2017a).

Main assumptions.

- A1 *Persistency of excitation:* The input must be persistently exciting of a sufficiently high order, see Willems et al. (2005).
- A2 *Bounded complexity model class:* An upper bound of the model order is given.

Assumption A1 is a standard one. Assumption A2 might be a mild hypothesis in some applications, but may also be quite critical elsewhere.

Being data-driven methods, the main input taken by the design functions is the set of I/O points. Based on such data, as already pointed out in the previous section, CbT is aimed to follow *a desired reference model*, whereas SLRA has the goal to track *a desired reference signal*.

A second input for the design function can therefore be either a model or a signal. In the former case, the SLRA method needs a reference signal. Such a signal is generated by the reference model. In the latter case, the CbT method needs a reference model. Such a model is obtained by analysis of the spectrum of the tracking signal.

A third compulsory input for the SLRA method is the bound on the model order. Notice also that, although CbT does not assume anything on the system dynamics, the controller structure needs to be fixed a-priori. This is not a strong requirement in many applications, e.g., in PID control, but may become a limitation in complex engineering tasks.

The output of the design will be a transfer function in the CbT case and the actual optimal control law in the SLRA method. This highlights a *fundamental difference* between the methods. The fact that the control law in SLRA is a function only of the identification dataset makes SLRA a *feedforward control* strategy, unlike CbT.

Tuning parameters/hyper-parameters. The fact that the two approaches are not bound to a particular model parameterization does not mean that they do not require tuning. The tuning parameters involved are sometimes referred to as hyper-parameters. The hyper-parameters have impact on the quality of the estimate of the optimal controller.

For the CbT method, the only hyper-parameter is the length of the instrumental variable $\zeta(t)$, i.e., l. Since l determines the size of the matrix describing the correlation between the matching error and the input, the higher the matrix the better the minimization process. Unfortunately, it has also been shown in Van Heusden et al. (2011) that the same parameter modulates the bias of the cost function. In other words, the lower l the lower the bias on the controller estimate. It follows that the choice of the best value of l is not a trivial task. In some papers, e.g. Formentin et al. (2014), it has been shown that a reasonable choice might be the length of the impulse response of M. Indeed, there is no formal proof for this statement.

For the SLRA method, the hyper-parameter is the order of the model. Overestimating the model order might lead to overfitting. In practice, however, the knowledge of an upper bound of the order is much less critical to obtain than an accurate parametric model of the system.

Applications. Since CbT needs to set the controller structure a-priori, such a method is suitable for applications where the controller parameterization is known (maybe because some controller is already running, e.g. in process control or servomechanisms) and one only needs a rapid (re-)calibration to optimize the parameters. This is useful for rapid prototyping or to compensate for aging effects in existing systems.

The intrinsic feedforward nature of the SLRA method makes it instead suitable for the open-loop generation of optimal references, e.g. in trajectory planning or model predictive control.

Pros and cons. The above comparison shows that the pros and cons of the methods are different, which means that the approaches should not be seen as competitors but as complementary methods.

First of all, when the parameterization of the controller is linear, CbT can be shown to be convex (see again the cost function to minimize (CC)). This is not the case for SLRA in any case of practical interest. On the other hand, in the SLRA method the controller does not need to be parameterized at all, as the controller output u_{f} (with no restrictions) is directly generated by the algorithm. As a side advantage, the controller corresponding to such an action could even be noncausal (thus not implementable following the other approach).

Another important feature of CbT is that the solution is independent of the reference signal. This means that the optimal controller will make the output follow the desired trajectory *whatever it is*. Instead, two different references would require two optimization runs in the SLRA method.

Concerning stability, the feedforward control obtained by the SLRA method obviously works only with systems that are already stable. However there is no risk to destabilize them. Instead, a bad choice of controller could even lead to system destabilization in CbT. To avoid such an issue, the data-driven stability constraint in Van Heusden et al. (2011) could be embedded within the CbT framework. Such a constraint allows one to guarantee the stability of the closed-loop system, but only asymptotically with the number of samples in the identification dataset. Since the size of a dataset is always limited, conservative choices are needed, sometimes leading to poor performance.

Finally, notice that, when the stability constraint is not needed, the minimizer of (CC) can be analytically found through least

Table 1. Overview of the comparison between noniterative CbT and structured low-rank approximation.

	CbT	SLRA
Assumptions	- LTI system - PE input	- LTI system - PE input - upper bound of the model order is known
Features	- feedback control *Inputs:* - I/O data - reference model - controller structure *Hyper-parameters:* length of IV *Output:* - controller transfer operator	- feedforward control *Inputs:* - I/O data - desired output - bound on the model order *Hyper-parameters:* bound on the model order *Output:* - control signal
Applications	- servo mechanisms - process control	- trajectory planning - MPC
Pros	- convex problem - solution is independent of the reference signal - stability guarantee for the closed-loop system	- no restrictions on the control signal - no need to parameterize the controller a-priori - non causal control
Cons	- linear controller parameterization - causal controller - only asymptotic stability guarantees - hard bounds on the control signal not allowed	- nonconvex problem - re-computation for new reference signal - batch method - non efficient computational methods

squares formulas. In the more general case, convex optimization tools can be used to return the solution (being both the cost function and the constraints convex functions of ρ). The computational burden of the SLRA method is instead higher, hence the method is not efficient if the number of parameters is high. We should stress however that such an increase in the computational load is due to a more flexible choice of the control action.

5. A NUMERICAL COMPARISON

The benchmark example of Landau et al. (1995) is used to compare the performance of the two methods. The plant is

$$G(q^{-1}) = \frac{0.28261q^{-3} + 0.50666q^{-4}}{A(z)}, \qquad (1)$$

where
$A(z) = 1 - 1.41833z^{-1} + 1.58939z^{-2} - 1.31608z^{-3} + 0.88642z^{-4}$.
Consider the feedback controller

$$K_o(q^{-1}) = \frac{\sum_{k=0}^{5} \rho_{0k} q^{-k}}{1 - q^{-1}}, \qquad (2)$$

where $\rho_o = [0.2045\ 0.2715\ 0.2931\ 0.1643\ 0.0084]^T$ and the reference model

$$M(q^{-1}) = \frac{G(q^{-1})K_o(q^{-1})}{1 + G(q^{-1})K_o(q^{-1})}. \qquad (3)$$

The objective of CbT is to design a controller of the form

$$K(q^{-1}) = \frac{\sum_{k=0}^{5} \rho_k q^{-k}}{1 - q^{-1}}. \qquad (4)$$

Notice that it is possible to realize in closed-loop the model reference M by a controller (4) (the optimal controller (2) is in fact included in this system class). The objective of the SLRA method is instead to design directly the input u_f. The validation is done by evaluating the (normalized) output tracking error

$$e := \|y_f - \widehat{y}_f\| / \|y_f\|$$

achieved by the control methods *on the true plant*.

In the simulation we use an output error setup with white noise and signal-to-noise ratio $\text{SNR} = \text{var}(y_0)/\sigma^2$ (with σ as a simulation parameter). The simulation horizon is $N = 500$ samples and 500 Monte Carlo experiments are run with the same input u and different realizations of the noise. For SLRA design, we use an initial approximation obtained from the unstructured low-rank approximation computed by the singular value decomposition. The tracking error for different values of σ^2 (and then for different SNR) is shown in Figure 1.

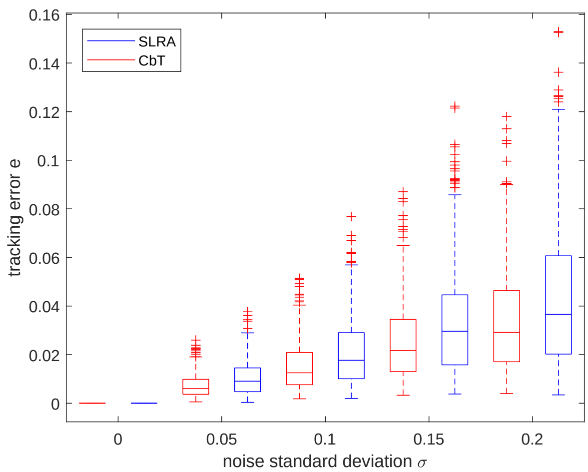

Fig. 1. In a Monte Carlo simulation (500 runs), the CbT method statistically outperforms the SLRA method.

Both in terms of mean tracking error and the variability of the results, the CbT method statistically outperforms the SLRA method. However, in the proposed example, the optimal controller *belongs to the controller set*, which is a strong assumption in real applications. If the set of controllers, or equivalently the reference model, is changed so that the selected controller class is not allowed to achieve a perfect closed-loop matching, the comparison gives less intuitive results.

Consider, e.g., the case where

$$M(q^{-1}) = \frac{0.1548q^{-3}}{1-1.213q^{-1}+0.3679q^{-2}}, \qquad (5)$$

which is unachievable for any choice of ρ in (4), as already shown in Campi et al. (2002). In such a case, the tracking error generated with the noiseless estimate (corresponding to a *bias error*, as the variability with respect to the noise realization is zero) is higher in the CbT case, as indicated in Table 2.

Table 2. Error e in case of zero noise and M in (5).

CbT	SLRA
$0.7672 \cdot 10^{-3}$	$0.0064 \cdot 10^{-3}$

When also the effect of noise is added (and the variance error overcomes the bias one), the trend becomes again that of Figure 1. However, the above observations highlight that CbT might not be the best choice when the SNR is high and the controller class is too simple compared to the dynamics of the process to control and the desired behaviour.

6. CONCLUSIONS

This paper compared the structured low-rank approximation and the correlation approaches for data-driven output tracking control. The main conclusion is that the two approaches are complementary, thus pros and cons depend on the problem at hand. For instance, the feedforward SLRA should be preferred when there are no stability issues, an upper bound on the model order is given, and there is little knowledge about the achievable performance in terms of output tracking. On the other hand, CbT is the right choice for unstable plants, online implementation, low SNR. Future work will compare the two approaches with other methods and investigate a possible way to merge them into a more effective unified approach to data-driven output tracking.

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