# Subspace, basis, and dimension

- $\mathscr{U} \subset \mathbb{R}^m$ is a subspace of a vector space $\mathbb{R}^m$ if $\mathscr{U}$ is a vector space

$$u, v \in \mathscr{U} \quad \Longrightarrow \quad \alpha u + \beta v \in \mathscr{U}, \quad \text{for all } \alpha, \beta \in \mathbb{R}$$

- The set $\{ u^{(1)}, \ldots, u^{(m)} \}$ is a basis of a vector space $\mathscr{U}$ if
  - $u^{(1)}, \ldots, u^{(m)}$ span $\mathscr{U}$, *i.e.*,

    $$\mathscr{U} = \text{span}(u^{(1)}, \ldots, u^{(m)}) := \{ \alpha_1 u^{(1)} + \cdots + \alpha_m u^{(m)} \mid \alpha_1, \ldots, \alpha_m \in \mathbb{R} \}$$

  - $\{ u^{(1)}, \ldots, v^{(m)} \}$ is an independent set of vectors.

- $\dim(\mathscr{U})$ — number of basis vectors (doen't depend on the basis)

# Null space of a matrix (kernel)

- kernel of $A$ — the set of vectors mapped to zero by $f(u) := Au$

$$\ker(A) := \{\, u \in \mathbb{R}^m \mid Au = 0 \,\}$$

- $y = A(u + v)$, for all $v \in \ker(A)$

  Interpretation: $\ker(A)$ is the uncertainty in finding $u$, given $y$.

  Interpretation: $\ker(A)$ is the freedom in the $u$'s that achieve $y$.

- $\ker(A) = \{\, 0 \,\} \quad \Longleftrightarrow \quad f(u) := Au$ is one-to-one

- $\ker(A) = \{\, 0 \,\} \quad \Longleftrightarrow \quad A$ is full column rank

# Range of a matrix (image)

- image of $A$ — the set of all vectors obtainable by $f(u) := Au$

$$\text{image}(A) := \{\, Au \mid u \in \mathbb{R}^{m} \,\}$$

- $\text{image}(A) =$ span of the columns of $A$
- $\text{image}(A) =$ set of vectors $y$ for which $Au = y$ has a solution
- $\text{image}(A) = \mathbb{R}^{p} \quad \Longleftrightarrow \quad f(u) := Au$ is onto ($\text{image}(f) = \mathbb{R}^{p}$)
- $\text{image}(A) = \mathbb{R}^{p} \quad \Longleftrightarrow \quad A$ is full row rank

# Change of basis

- standard basis vectors in $\mathbb{R}^{\mathrm{m}}$ — the columns $e^{(1)}, \ldots, e^{(\mathrm{m})}$ of $I_{\mathrm{m}}$
- Elements of $u \in \mathbb{R}^{\mathrm{m}}$ are coordinates of $x$ w.r.t. standard basis.
- A new bases is given by the columns $v^{(1)}, \ldots, v^{(\mathrm{m})}$ of $V \in \mathbb{R}^{\mathrm{m} \times \mathrm{m}}$.
- The coordinates of $u$ in the new basis are $\widetilde{u}_1, \ldots, \widetilde{u}_{\mathrm{m}}$, such that

$$u = \widetilde{u}_1 v^{(1)} + \cdots + \widetilde{u}_{\mathrm{m}} v^{(\mathrm{m})} = V\widetilde{u} \quad \Longrightarrow \quad \widetilde{u} = V^{-1} u$$

- $V^{-1}$ transforms standard basis coordinates $u$ into $V$-coordinates

# Similarity transformation

- Consider linear operator $f : \mathbb{R}^m \to \mathbb{R}^m$, given by $f(u) = Au$, $A \in \mathbb{R}^{m \times m}$.
- Change standard basis to basis defined by columns of $V \in \mathbb{R}^{m \times m}$.
- The matrix representation of $f$ changes to $V^{-1}AV$:

$$u = V\widetilde{u}, \quad y = V\widetilde{y} \qquad \Longrightarrow \qquad \widetilde{y} = (V^{-1}AV)\widetilde{u}$$

- $A \mapsto V^{-1}AV$ — similarity transformation of $A$