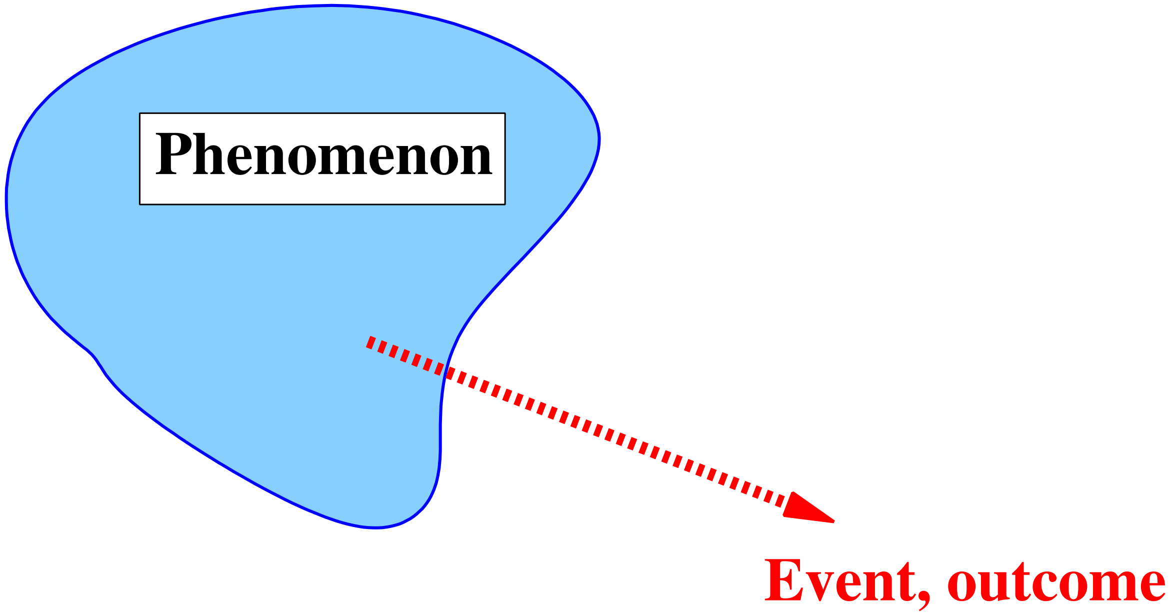
Modeling

Suppose that we have a 'real' phenomenon.

The phenomenon produces 'events' (synonym: 'outcomes').



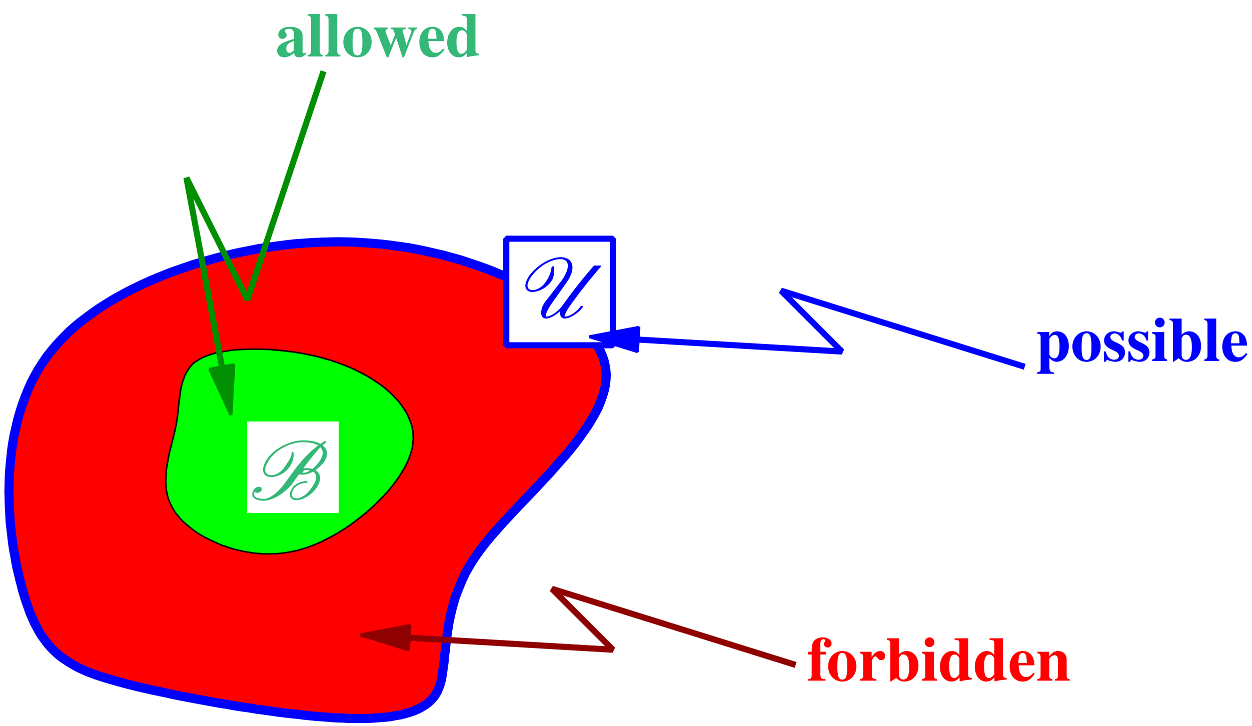
We view a (deterministic) model for the phenomenon as a prescription of which events can occur,
and which events cannot occur.

The behavior

A *mathematical model* $:\Leftrightarrow$ **a pair** $(\mathscr{U}, \mathscr{B})$ **with**

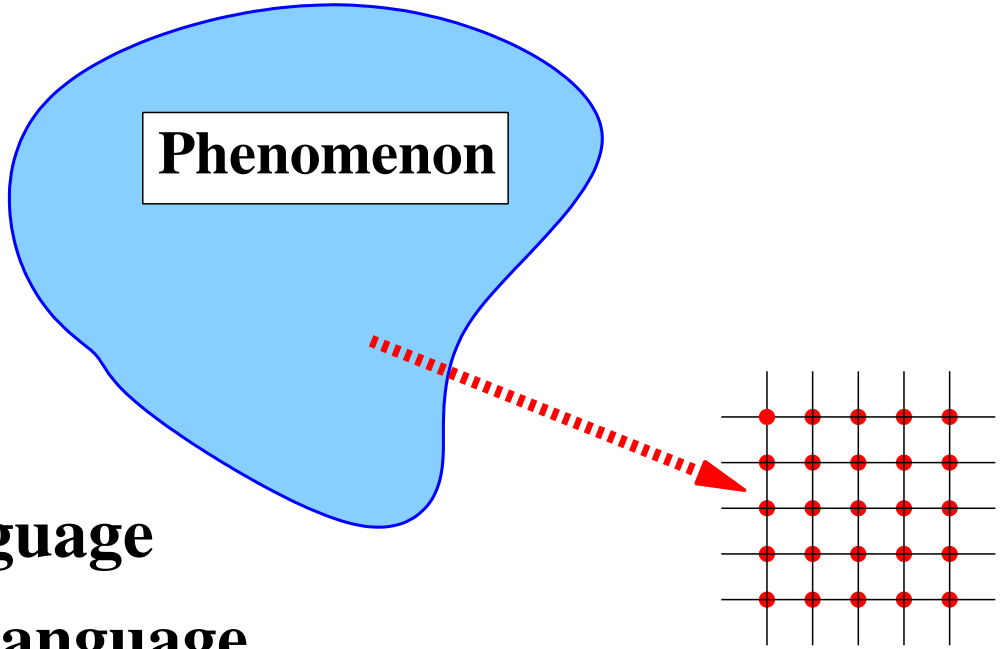
$\mathscr{U}$ **the universum of events**

$\mathscr{B} \subseteq \mathscr{U}$ **the behavior of the model**



Discrete event phenomena

If $\mathscr{U}$ is a finite set, or strings of elements from a finite set, we speak about discrete event systems (DESs).



Examples:

- **Words in a natural language**
- **Sentences in a natural language**
- **DNA sequences**
- **LaTeX code**
- **Block codes**

Discrete event phenomena

- **Words in a natural language**

$\mathscr{U} = \mathbb{A}^*$ **(:= all finite strings with letters from $\mathbb{A}$)**
with $\mathbb{A} = \{a, \ldots, z, A, \ldots, Z\}$.

$\mathscr{B}$ **= all words recognized by the spelling checker,**
for example, behavior $\in \mathscr{B}$, SPQR $\notin \mathscr{B}$.
$\mathscr{B}$ **is basically specified by enumeration.**

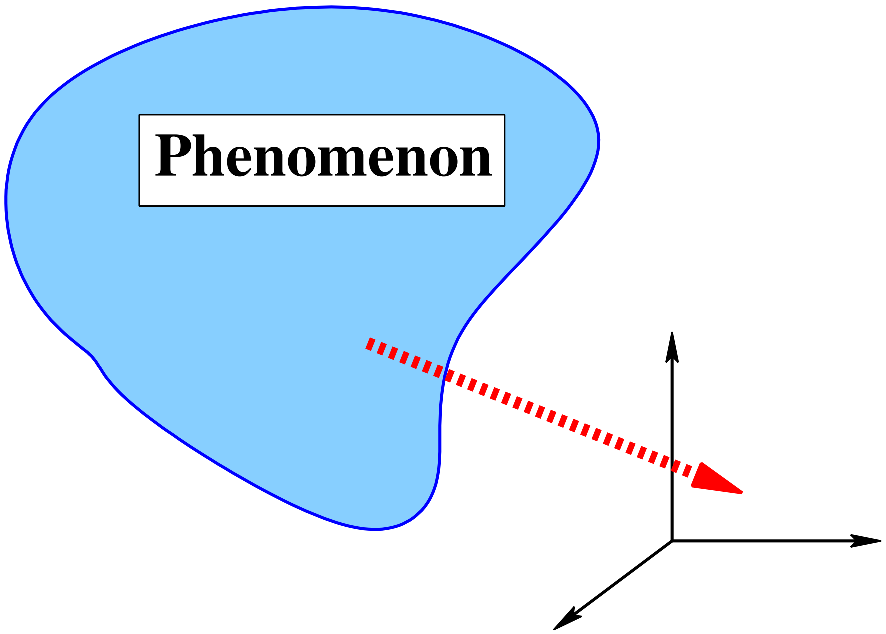
- **Sentences in a natural language**

$\mathscr{U} = \mathbb{A}^*$ **(:= all finite strings with letters from $\mathbb{A}$)**
with $\mathbb{A} = \{a, \ldots, z, A, \ldots, Z, \ , .;: \text{‘’”}' - ()!?, \ \textbf{etc.}\}$.

$\mathscr{B}$ **= all legal sentences.**
Specifying $\mathscr{B}$ is a complicated matter, involving grammars.

Continuous phenomena

If $\mathscr{U}$ is a (subset of a) finite-dimensional real (or complex) vector space, we speak about **continuous models.**



Examples:

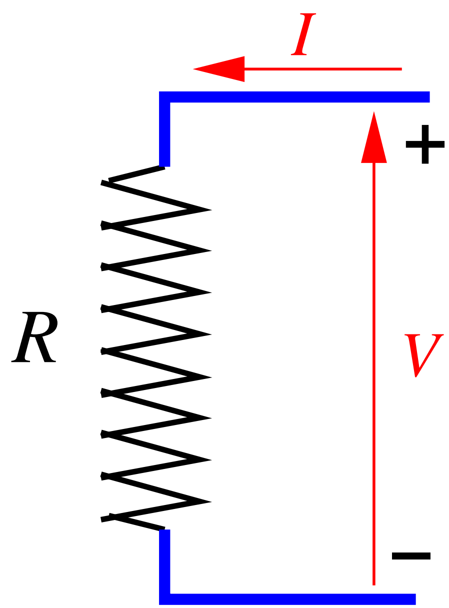
- The gas law
- A spring
- The gravitational attraction of two bodies
- A resistor

Continuous phenomena

- **A resistor**

Event: voltage V, current I.

Throughout, we take the current positive when it runs *into* the circuit, and we take the voltage positive when it goes *from higher to lower* potential.



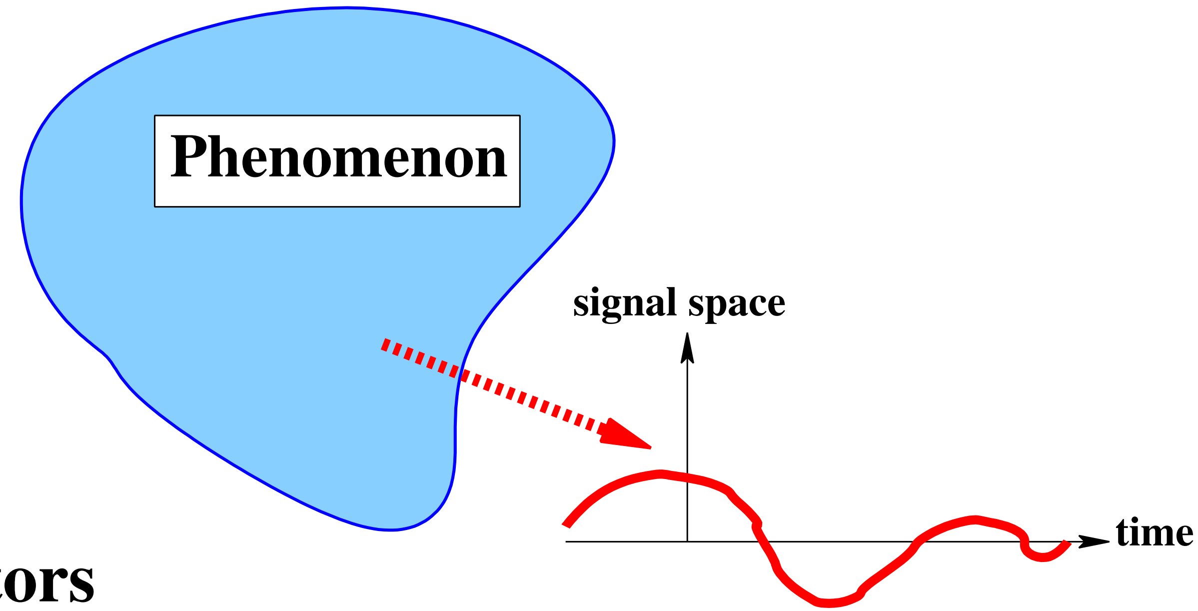
Georg Ohm (1789–1854)

$\mathscr{U} = \mathbb{R} \times \mathbb{R}$;

$\mathscr{B} = \{(V,I) \in \mathbb{R} \times \mathbb{R} \mid V = RI\}$ **(Ohm's law).**

Dynamical phenomena

If $\mathscr{U}$ is a set of functions of time, we speak about dynamical models.



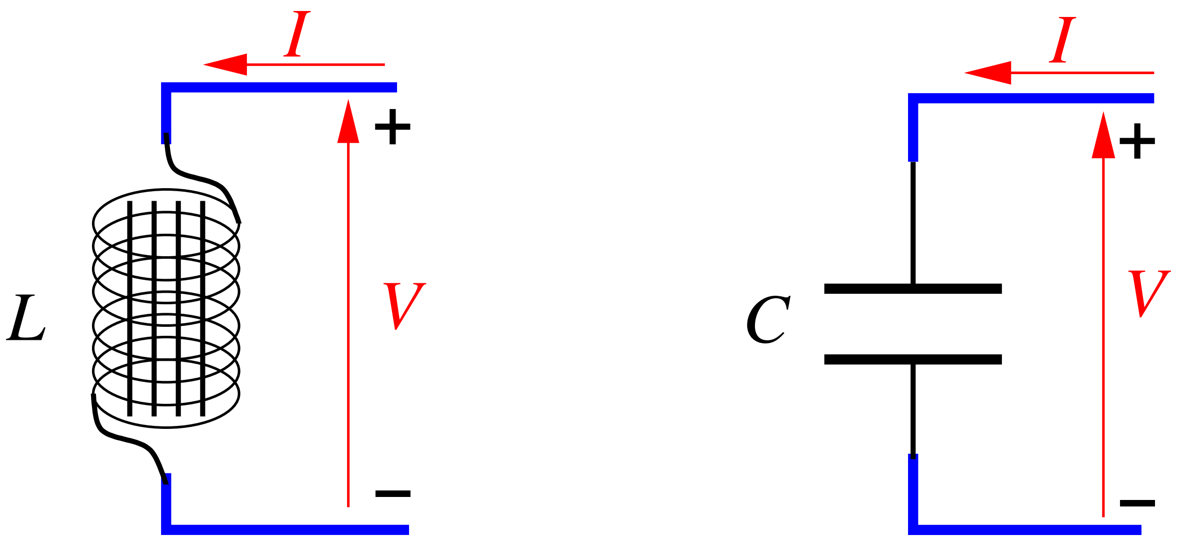
Examples:

- **Inductors, capacitors**
- **Kepler's laws**
- **Newton's second law**
- **Convolutional codes**

Dynamical phenomena

▶ Inductors and capacitors

Event: voltage and current as a function of time.



$$\mathscr{U} = (\mathbb{R} \times \mathbb{R})^{\mathbb{R}};$$

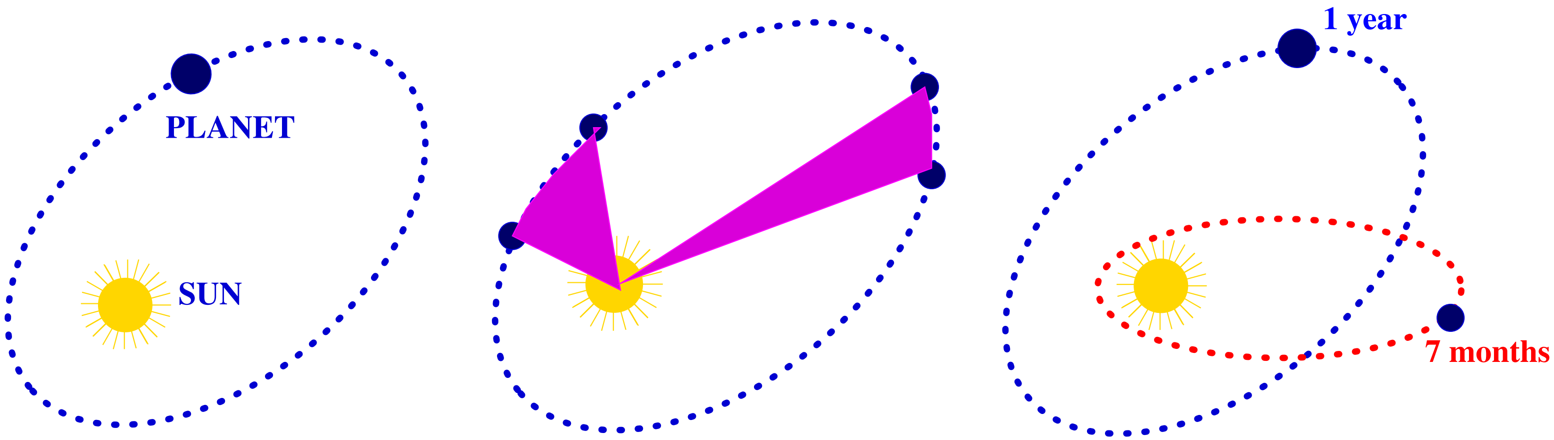
$$\mathscr{B} = \{(V, I) : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \mid L\tfrac{d}{dt}I = V\}$$ **(inductor),**

$$\mathscr{B} = \{(V, I) : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \mid C\tfrac{d}{dt}V = I\}$$ **(capacitor).**

Dynamical phenomena

- **Kepler's laws**

Event: the position of a planet as a function of time.



K1: ellipse, sun in focus,
K2: equal areas in equal times,
K3: square of the period
= third power of major axis.

$\mathcal{U} = (\mathbb{R}^3)^{\mathbb{R}}$;

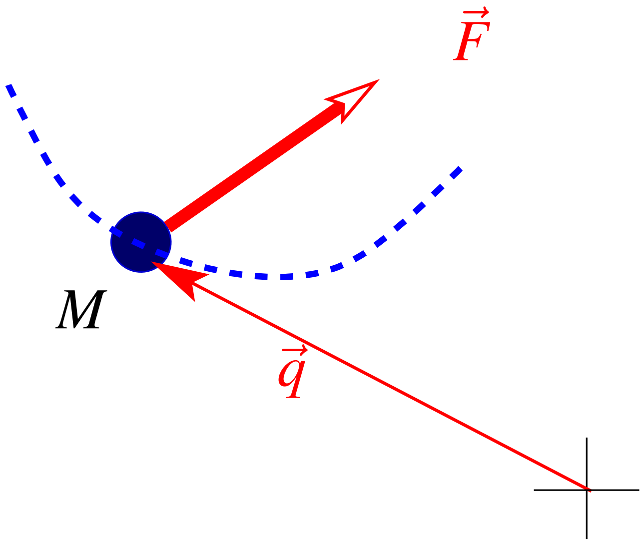
$\mathcal{B} = \{\vec{q} : \mathbb{R} \to \mathbb{R}^3 \mid$ **K1, K2, & K3 hold** $\}$.

Johannes Kepler (1571–1630)

Dynamical phenomena

- **<u>Newton's second law</u>**

  <u>Event</u>: the position of a pointmass and the force acting on it, both as a function of time.



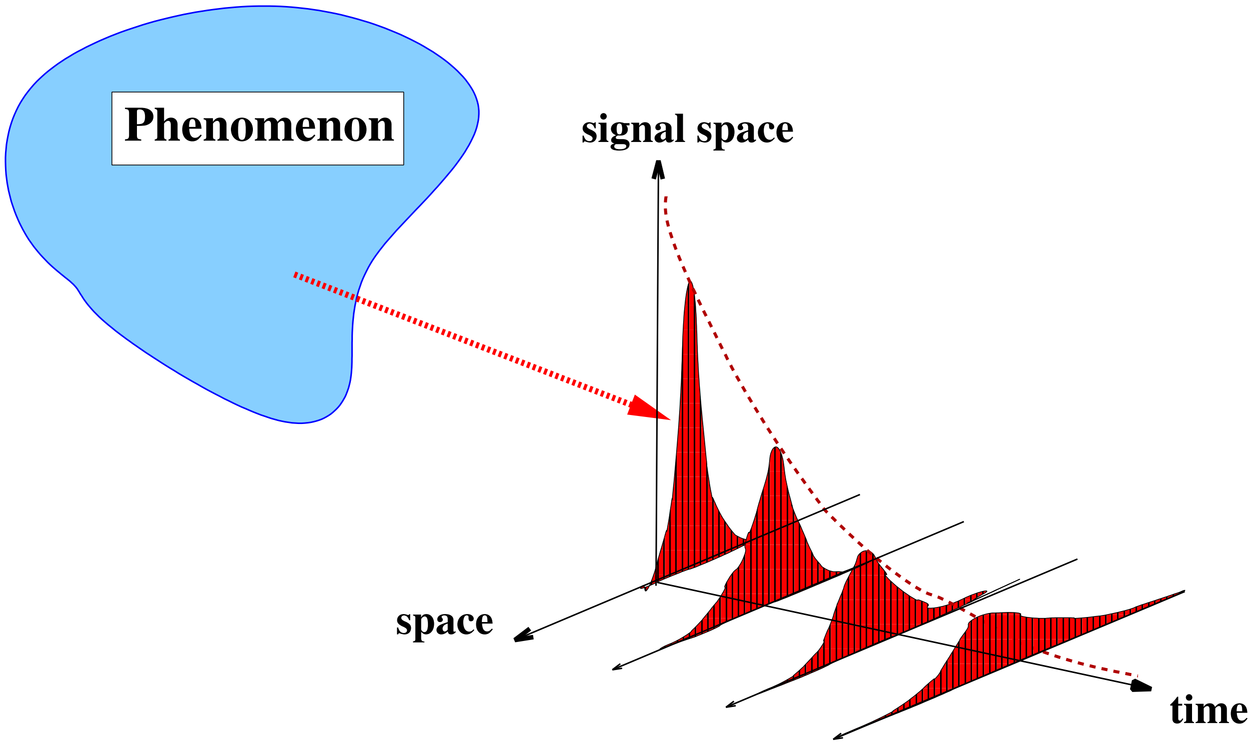
Newton painted by William Blake

$$\mathscr{U} = (\mathbb{R}^3 \times \mathbb{R}^3)^{\mathbb{R}};$$

$$\mathscr{B} = \{(\vec{q}, \vec{F}) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \mid \vec{F} = M\frac{d^2}{dt^2}\vec{q}\}.$$

Distributed phenomena

If $\mathscr{U}$ is a set of functions of space and time, we speak about distributed parameter systems.



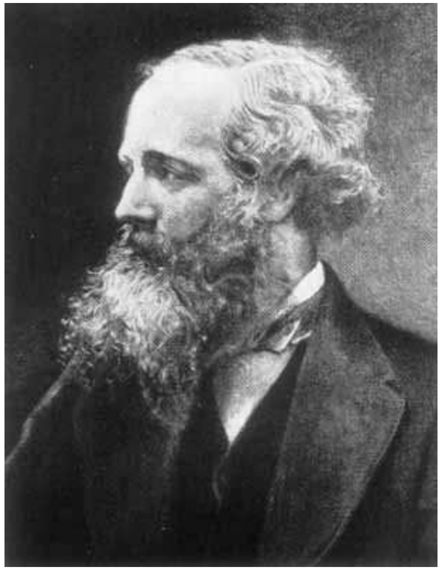
Examples:

- **Heat diffusion**
- **Wave equation**
- **Maxwell's equations**

Distributed phenomena

- **Maxwell's equations**

  Event: electric field, magnetic field, current density, charge density as a function of time and space.



James Clerk Maxwell (1831–1879)

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho \,, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\ \nabla \cdot \vec{B} &= 0 \,, \\ c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}. \end{aligned}$$

$$\mathscr{U} = (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})^{\mathbb{R}^4};$$

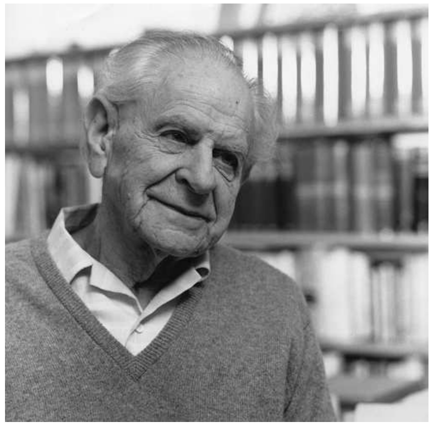
$$\mathscr{B} = \{(\vec{E}, \vec{B}, \vec{j}, \rho) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \mid \text{Maxwell's equations are satisfied}\}.$$

Behavioral models

The behavior captures the essence of what a model is.

The behavior is all there is.
Equivalence of models, properties of models,
symmetries, system identification, etc.
must all refer to the behavior.

Every 'good' scientific theory is prohibition:
it forbids certain things to happen.
The more it forbids, the better it is.

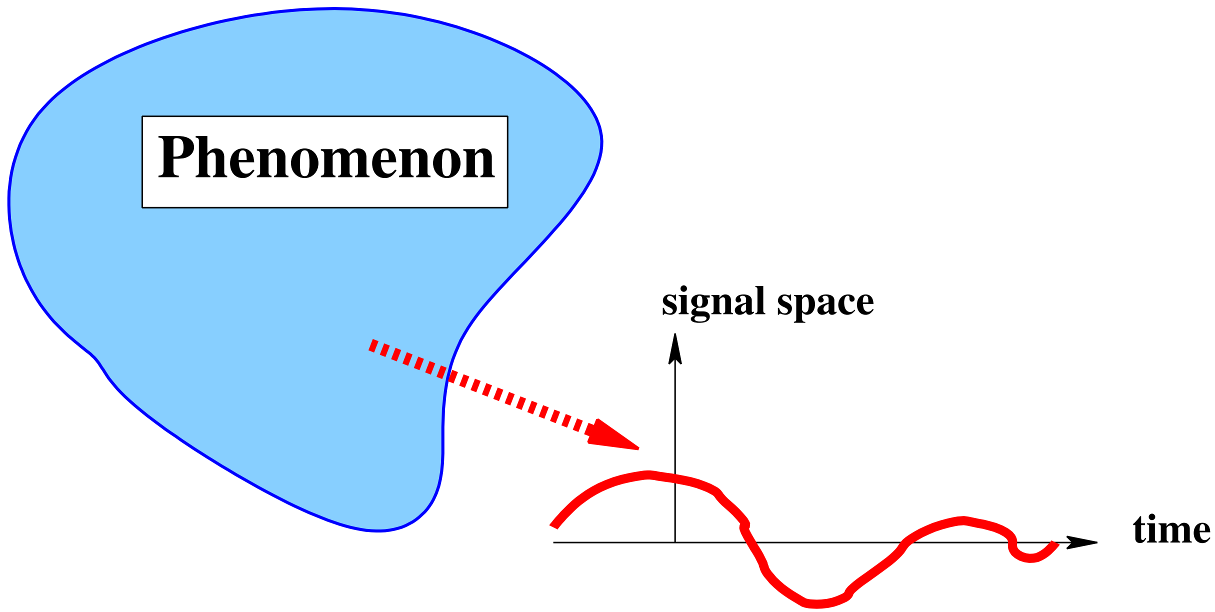


Karl Popper (1902-1994)

Replace 'scientific theory' by 'mathematical model'.

The dynamic behavior

In dynamical systems, the ‘events’ are maps, with the time-axis as domain. The events are functions of time.



It is convenient to distinguish, in the notation,

the domain of the event maps, the **time set,**

and the codomain, the **signal space,**

that is, the set where the functions take on their values.

The dynamic behavior

Definition: A **dynamical system** $:\Leftrightarrow$ $(\mathbb{T}, \mathbb{W}, \mathscr{B})$, **with**

- $\mathbb{T} \subseteq \mathbb{R}$ **the time set,**
- $\mathbb{W}$ **the signal space,**
- $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the behavior,**
  that is, $\mathscr{B}$ **is a family of maps from** $\mathbb{T}$ **to** $\mathbb{W}$**.**

$w : \mathbb{T} \to \mathbb{W} \in \mathscr{B}$ **means:** **the model allows the trajectory** w**,**

$w : \mathbb{T} \to \mathbb{W} \notin \mathscr{B}$ **means:** **the model forbids the trajectory** w**.**

The dynamic behavior

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$w : \mathbb{T} \to \mathbb{W} \notin \mathscr{B}$ **means: the model forbids the trajectory** w**.**

Mostly, $\mathbb{T} = \mathbb{R}, \mathbb{R}_+ := [0, \infty), \mathbb{Z}$**, or** $\mathbb{N} := \{0, 1, 2, \ldots\}$**,**

$\mathbb{W} =$ **(a subset of)** $\mathbb{R}^{\mathtt{w}}$**, for some** $\mathtt{w} \in \mathbb{N}$**,**

$\mathscr{B}$ **is then a family of trajectories taking values in a finite-dimensional real vector space.**

$\mathbb{T} = \mathbb{R}$ **or** $\mathbb{R}_+ \rightsquigarrow$ **'continuous-time' systems,**

$\mathbb{T} = \mathbb{Z}$ **or** $\mathbb{N} \rightsquigarrow$ **'discrete-time' systems.**

Dynamical systems described by differential equations

Consider the ODE

$$f\left(w, \frac{d}{dt}w, \frac{d^2}{dt^2}w, \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w\right) = 0, \qquad (*)$$

with

$$f : \mathbb{W} \times \underbrace{\mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{w}} \times \cdots \times \mathbb{R}^{\mathtt{w}}}_{\mathtt{n}\ \textbf{times}} \to \mathbb{R}^{\bullet}, \qquad \mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}.$$

Some may prefer to write

$$f \circ \left(w, \frac{d}{dt}w, \frac{d^2}{dt^2}w, \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w\right) = 0,$$

instead of $(*)$, but we leave the $\circ$ notation to puritans.

Linearity and time-invariance

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ is said to be

linear :$\Leftrightarrow$

$\mathbb{W}$ **is a vector space (over a field $\mathbb{F}$) and**

$[\![w_1, w_2 \in \mathscr{B}$ **and** $\alpha \in \mathbb{F}]\!] \Rightarrow [\![w_1 + \alpha w_2 \in \mathscr{B}]\!]$.

Linearity $\Leftrightarrow$ the 'superposition principle' holds.

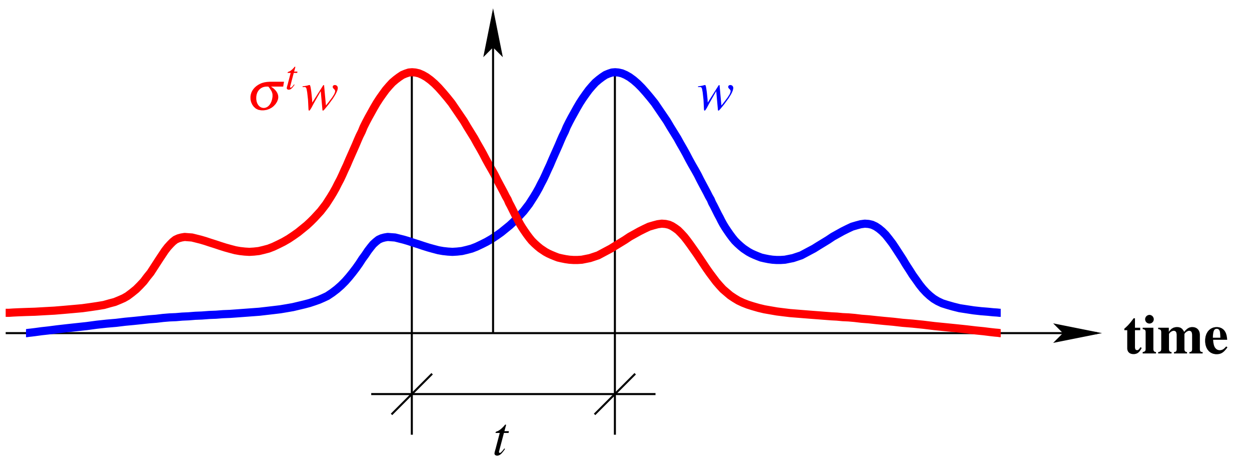
Linearity and time-invariance

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ is said to be

time-invariant $:\Leftrightarrow \mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z},$ **or** $\mathbb{N}$**, and**
$[\![w \in \mathscr{B}$ **and** $t \in \mathbb{T}]\!] \Rightarrow [\![\sigma^t w \in \mathscr{B}]\!]$**.**

σ^t **denotes the backwards t-shift, defined as**

$$\sigma^t w : \mathbb{T} \to \mathbb{W}, \qquad \sigma^t w(t') := w(t'+t).$$



Time-invariance $\Leftrightarrow$ shifts of 'legal' trajectories are 'legal'.

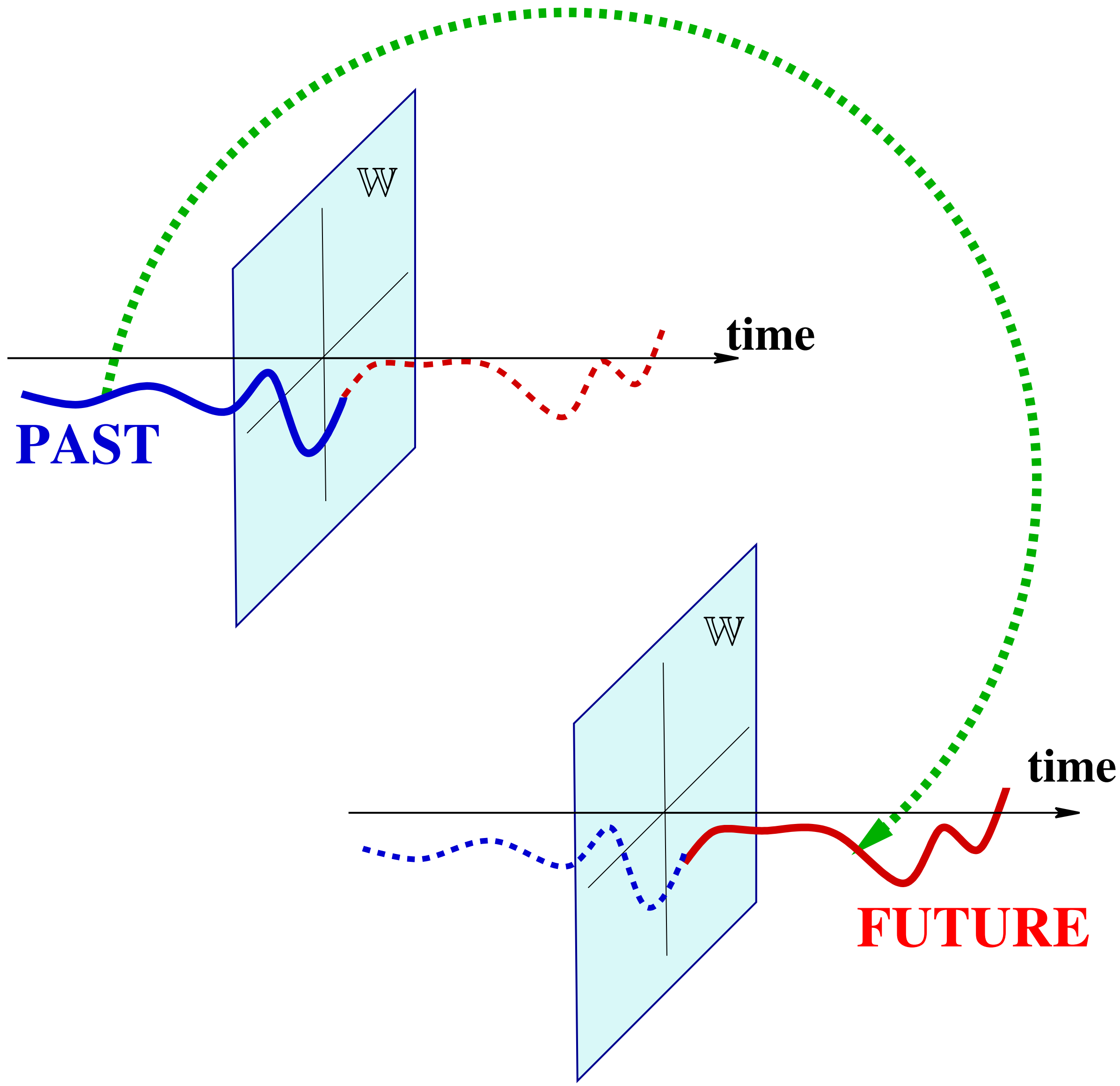
Autonomous systems

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$**, with** $\mathbb{T} = \mathbb{R}$ **or** $\mathbb{Z}$**, is said to be**

autonomous $:\Leftrightarrow$

$$[\![w_1, w_2 \in \mathscr{B}, \textbf{ and } w_1(t) = w_2(t) \textbf{ for } t < 0]\!] \Rightarrow [\![w_1 = w_2]\!].$$

Autonomous in a picture



autonomous :⇔ the past implies the future.

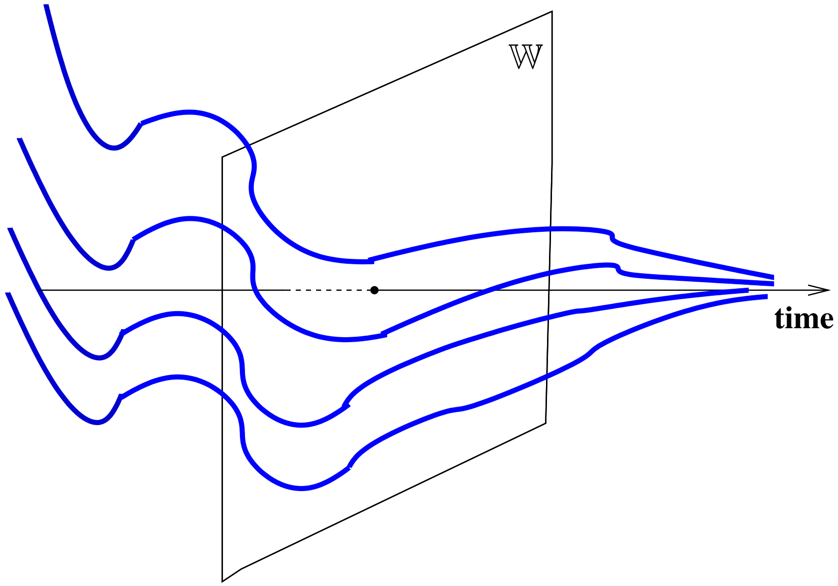
Stability

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$**, with** $\mathbb{T} = \mathbb{R}, [0, \infty),$ $\mathbb{Z},$ **or** $\mathbb{N}$**, and** $\mathbb{W}$ **a normed vector space (for simplicity), is said to be stable** $:\Leftrightarrow [\![w \in \mathscr{B}]\!] \Rightarrow [\![w(t) \to 0$ **for** $t \to \infty]\!]$.

Stability

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$, with $\mathbb{T} = \mathbb{R}, [0, \infty)$, $\mathbb{Z}$, or $\mathbb{N}$, and $\mathbb{W}$ a normed vector space (for simplicity), is said to be stable $:\Leftrightarrow [\![w \in \mathscr{B}]\!] \Rightarrow [\![w(t) \rightarrow 0$ for $t \rightarrow \infty]\!]$.

In a picture



stability $:\Leftrightarrow$ all trajectories go to 0.

Sometimes this is referred to as 'asymptotic stability'.
There exist numerous other stability concepts for dynamical systems!

Controllability

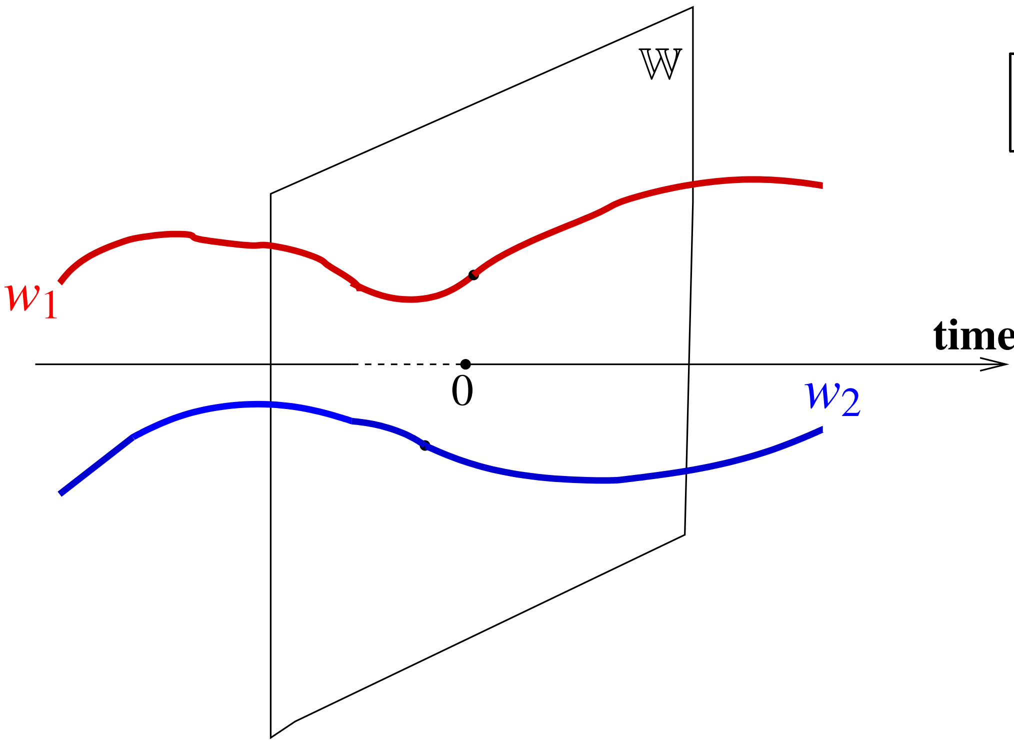
The time-invariant (to avoid irrelevant complications) dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, is said to be

controllable $:\Leftrightarrow$

for all $w_1, w_2 \in \mathscr{B}$,
there exist $T \in \mathbb{T}, T \geq 0$, **and** $w \in \mathscr{B}$,
such that

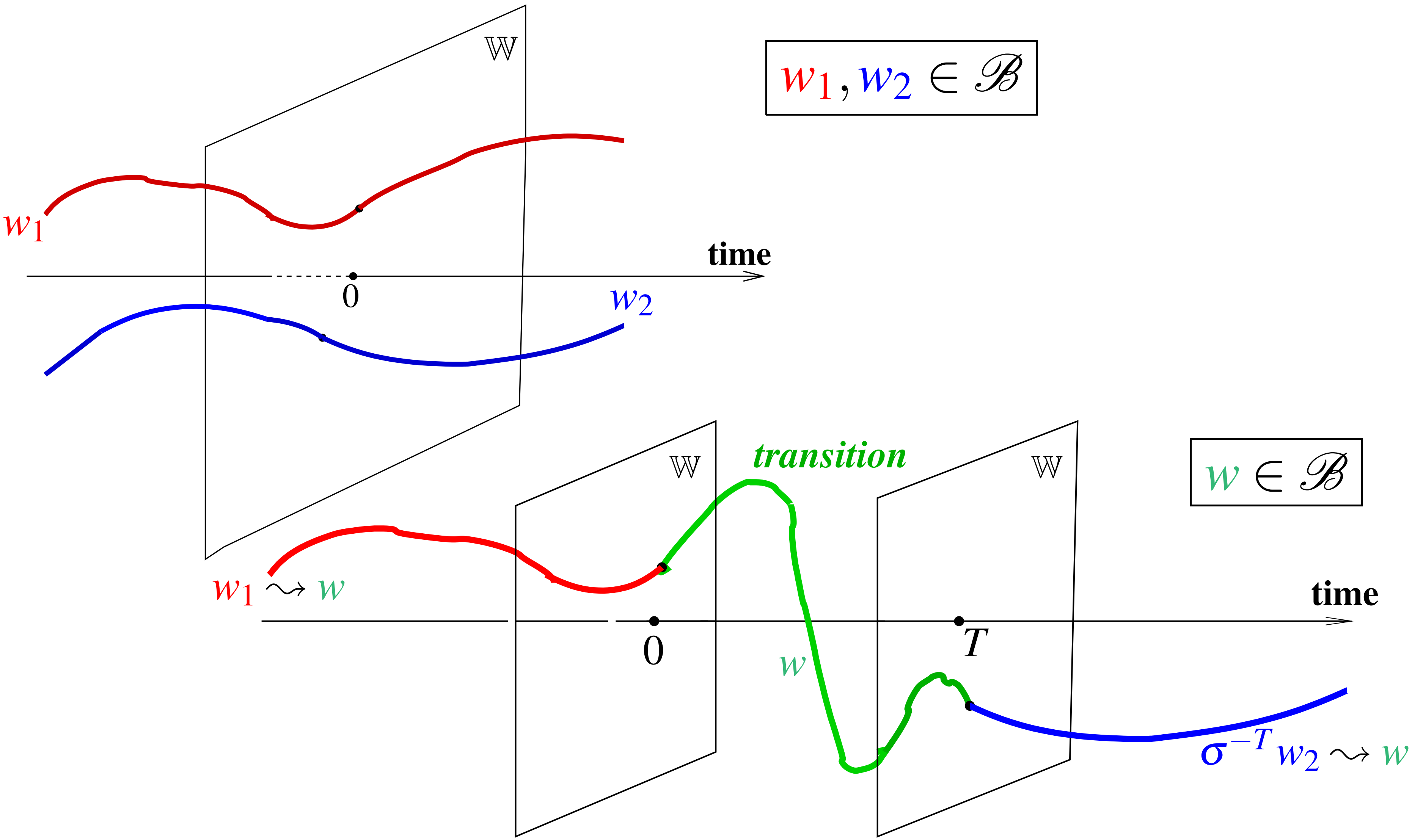
$$w(t) = \begin{cases} w_1(t) & \textbf{for } t < 0; \\ w_2(t-T) & \textbf{for } t \geq T. \end{cases}$$

Controllability in a picture



$w_1, w_2 \in \mathscr{B}$

Controllability in a picture



controllability :⇔ concatenability of trajectories after a delay

Stabilizability

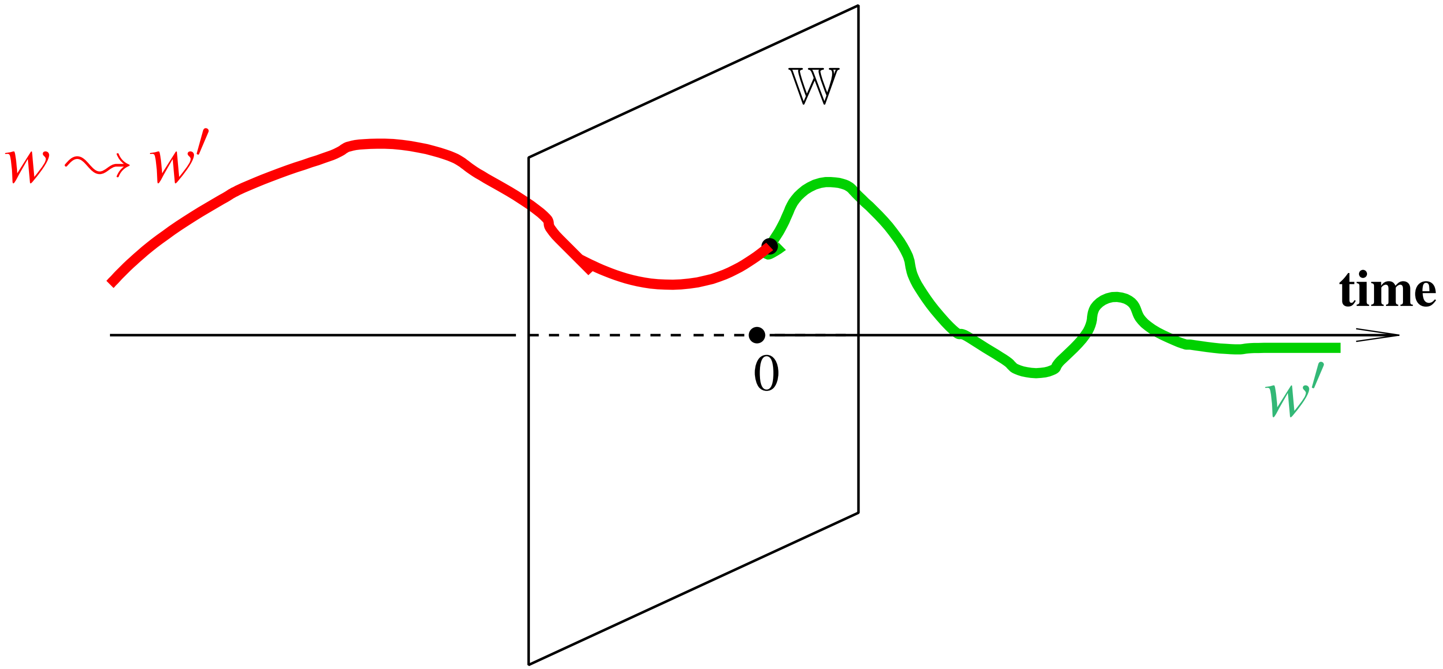
The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and $\mathbb{W}$ a normed vector space (for simplicity), is said to be

stabilizable $:\Leftrightarrow$ **for all** $w \in \mathscr{B}$**, there exist** $w' \in \mathscr{B}$**, such that**

$$w'(t) = w(t) \quad \textbf{for } t < 0,$$

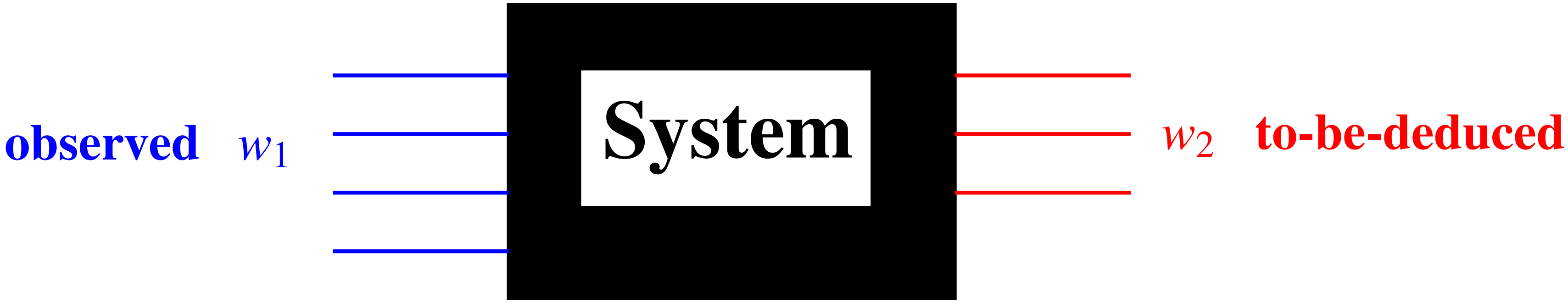
$$w'(t) \longrightarrow 0 \quad \textbf{for } t \longrightarrow \infty.$$

Stabilizability in a picture



stabilizability :$\Leftrightarrow$ all trajectories can be steered to 0.

Observability

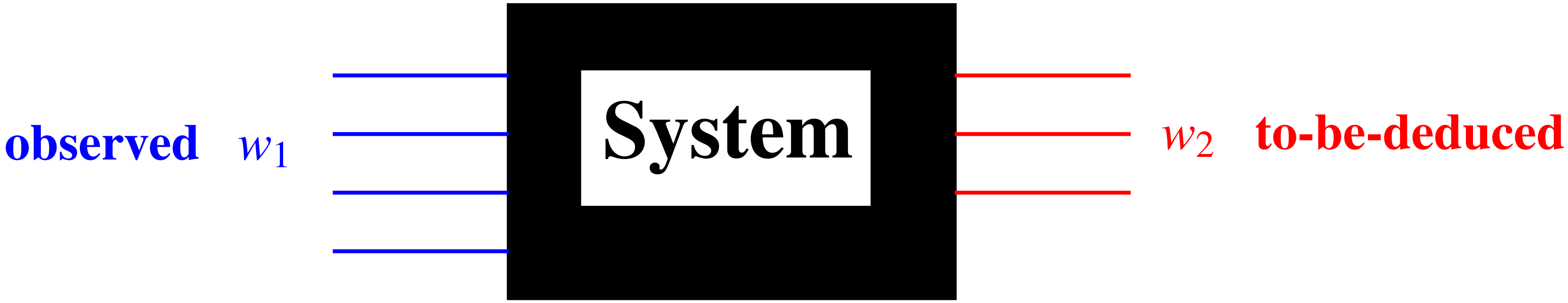


Consider the dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathscr{B})$**.**

w_2 **is said to be observable from** w_1 **in** $\Sigma :\Leftrightarrow$

$$[\![(w_1, w_2), (w_1', w_2') \in \mathscr{B} \text{ and } w_1 = w_1']\!] \Rightarrow [\![w_2 = w_2']\!].$$

Observability



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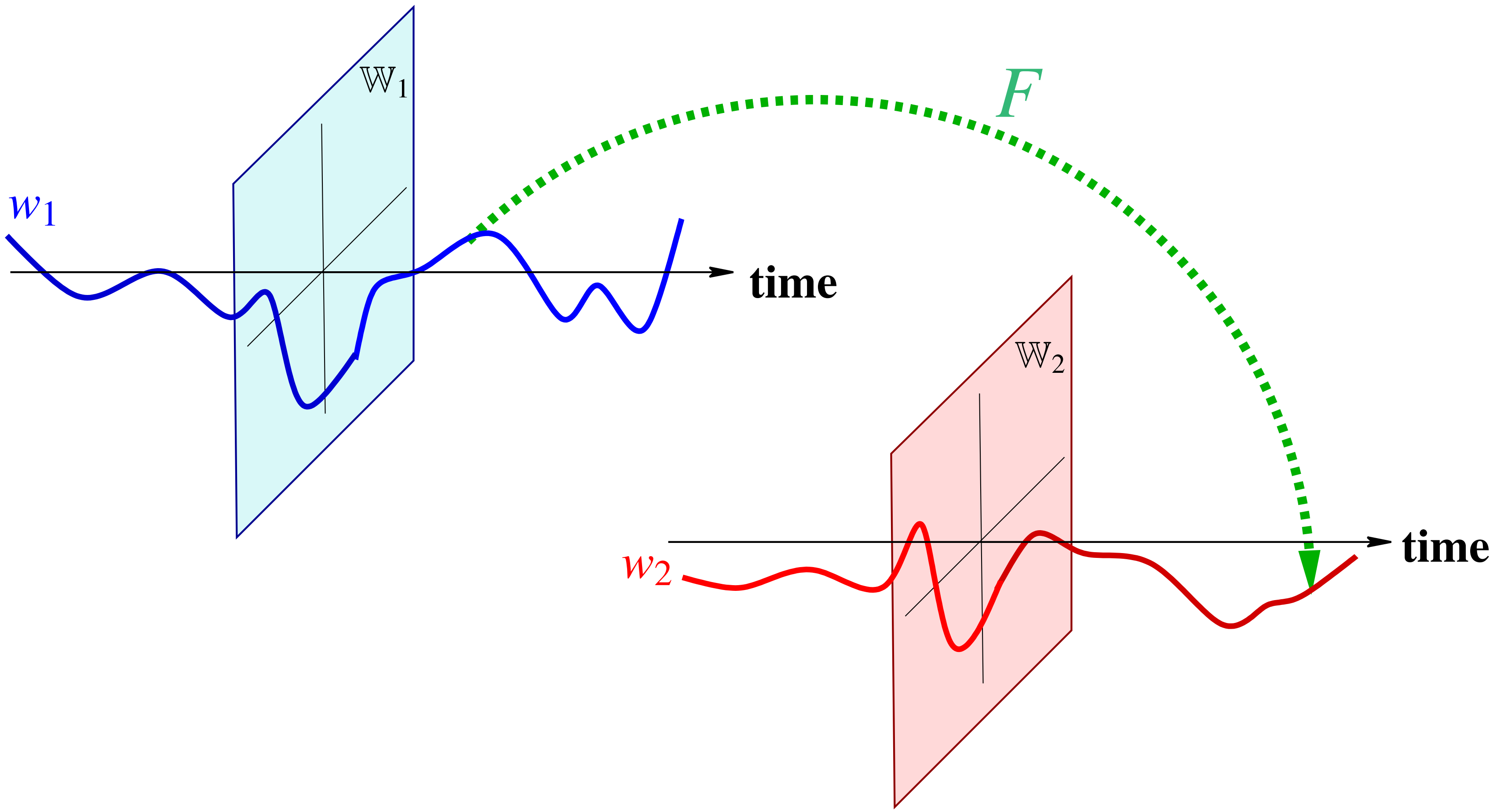
w_2 **is said to be observable from** w_1 **in** Σ $:\Leftrightarrow$

$$[\![(w_1, w_2), (w_1', w_2') \in \mathscr{B} \text{ and } w_1 = w_1']\!] \Rightarrow [\![w_2 = w_2']\!].$$

Observability $:\Leftrightarrow$ w_2 **may be deduced from** w_1.

!!! Knowing the laws of the system !!!

Observability in a picture



Equivalently, there exists a map $F : \mathbb{W}_1^{\mathbb{T}} \to \mathbb{W}_2^{\mathbb{T}}$**, such that**

$$[\![(w_1, w_2) \in \mathscr{B}]\!] \Rightarrow [\![w_2 = F(w_1)]\!].$$

Detectability

Consider the dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathscr{B})$**,**
with $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z}$**, or** $\mathbb{N}$**,**
and $\mathbb{W}$ **a normed vector space (for simplicity).**

w_2 **is said to be detectable from** w_1 **in** $\Sigma :\Leftrightarrow$

$$[\![(w_1, w_2), (w_1', w_2') \in \mathscr{B} \text{ and } w_1 = w_1']\!] \Rightarrow [\![w_2(t) - w_2'(t) \to 0 \quad \text{for } t \to \infty]\!].$$

Detectability $:\Leftrightarrow$ w_2 **can be asymptotically deduced from** w_1**.**

State equations

We now discuss how state models fit in.

$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u), \quad w = \begin{bmatrix} u \\ y \end{bmatrix}, \qquad (\spadesuit)$$

with $u : \mathbb{R} \to \mathbb{U}$ **the input, $y : \mathbb{R} \to \mathbb{Y}$ the output, and $x : \mathbb{R} \to \mathbb{X}$ the state.**

In particular, the linear case, these systems are parametrized by the 4 matrices $(A,B,C,D) \rightsquigarrow$

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$**.**
These models have dominated linear system theory since the 1960's.

State equations

We now discuss how state models fit in.

$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u), \quad w = \begin{bmatrix} u \\ y \end{bmatrix}, \qquad (\spadesuit)$$

with $u : \mathbb{R} \to \mathbb{U}$ **the input, $y : \mathbb{R} \to \mathbb{Y}$ the output, and** $x : \mathbb{R} \to \mathbb{X}$ **the state.**

It is common to view state space systems as models to describe the input/output behavior by means of input/state/output equations, with the state as latent variable. Define

$$\mathscr{B}_{\text{extended}} := \{(u,y,x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \mid (\spadesuit) \textbf{ holds}\},$$

$$\mathscr{B} := \{(u,y) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \mid \exists\, x : \mathbb{R} \to \mathbb{X} \textbf{ such that } (\spadesuit) \textbf{ holds}\}.$$

State controllability

State models propagated in the 1960's under the influence of R.E. Kalman.

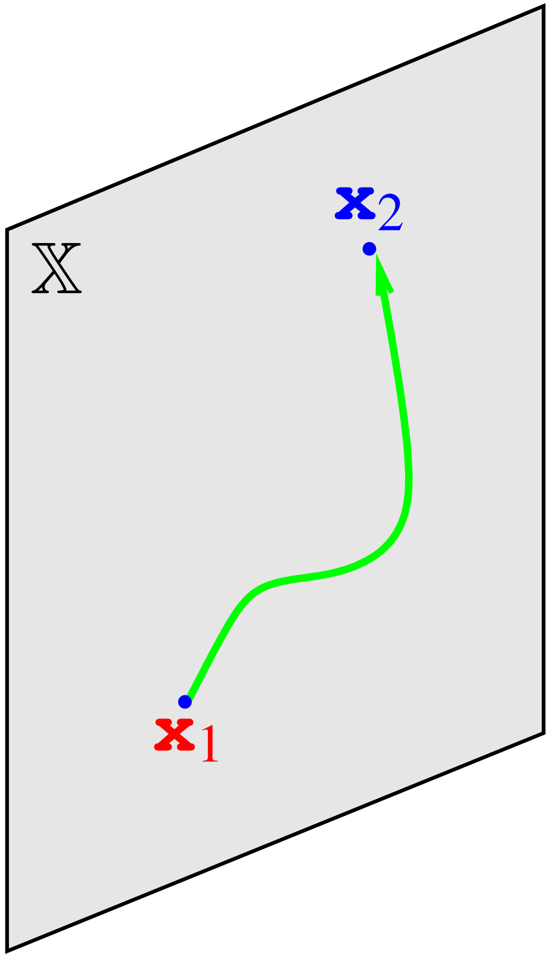
Especially important in this development were the notions of state controllability and state observability.

Rudolf Kalman (1930–)

State controllability

(♠) is said to be **state controllable** if for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$, there exists $T \geq 0$, $u : \mathbb{R} \to \mathbb{U}$, and $x : \mathbb{R} \to \mathbb{X}$, such that

1. $\frac{d}{dt}x(t) = f(x(t), u(t))$ for $0 \leq t \leq T$,
2. $x(0) = \mathbf{x}_1$,
3. $x(T) = \mathbf{x}_2$.



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1. $\frac{d}{dt}x(t) = f(x(t), u(t))$ **for** $0 \leq t \leq T$,
2. $x(0) = \mathbf{x}_1$,
3. $x(T) = \mathbf{x}_2$.

It is easy to prove that
⟦state controllability⟧
$\Leftrightarrow$ **⟦behavioral controllability of $\mathscr{B}_{\text{extended}}$⟧.**
⟦state controllability⟧ $\Rightarrow$ ⟦behavioral controllability of $\mathscr{B}$⟧.

Behavioral controllability makes controllability into a genuine, an intrinsic, representation independent system property.

State observability

(♠) is said to be state observable if

$$[\![(u,y,x_1),(u,y,x_2)\in\mathscr{B}_{\text{extended}}]\!] \Rightarrow [\![x_1(0)=x_2(0)]\!].$$

State observability

(♠) is said to be **state observable** if

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It is easy to prove that
$[\![$**state observability**$]\!]\Leftrightarrow[\![$**behavioral observability of** $\mathscr{B}_{\text{extended}}]\!]$,
with (u,y) **as 'observed' variables, and** x **as 'to-be-deduced' variable.**

State observability

(♠) is said to be **state observable** if

$$[\![(u,y,x_1),(u,y,x_2)\in\mathscr{B}_{\text{extended}}]\!] \Rightarrow [\![x_1(0)=x_2(0)]\!].$$

It is easy to prove that
$[\![$state observability$]\!]$ $\Leftrightarrow$ $[\![$behavioral observability of $\mathscr{B}_{\text{extended}}]\!]$, with (u,y) as 'observed' variables, and x as 'to-be-deduced' variable.

Behavioral controllability and observability are meaningful generalizations of state controllability and observability.

Why should we be so interested in the state?

Summary

- **A phenomenon produces 'events', 'outcomes'.**
  $\rightsquigarrow$ the universum of events $\mathscr{U}$.
- **A *mathematical model* specifies a subset $\mathscr{B}$ of $\mathscr{U}$.**
  $\mathscr{B}$ is the behavior and specifies which events can occur, according to the model.
- **In dynamical systems, the events are maps from the time set to the signal space.**
- **Controllability, observability, and similar properties can be nicely defined within this setting.**
- **State models are a more structured class of dynamical systems.**