Dynamical system

The set of functions (signals) $w : \mathbb{T} \to \mathbb{W}$ from $\mathbb{T}$ to $\mathbb{W}$ is denoted by $\mathbb{W}^{\mathbb{T}}$.

- $\mathbb{W}$ — variable space
- $\mathbb{T} \subset \mathbb{R}$ — time axis
- $\mathbb{W}^{\mathbb{T}}$ — trajectory space

A dynamical system $\mathscr{B} \subset \mathbb{W}^{\mathbb{T}}$ is a set of trajectories (a behaviour).

$w \in \mathscr{B}$ means that w is a possible trajectory of the system $\mathscr{B}$

Note: the set definition is extremely general (and therefore abstarct). For example, it is not specialized to linear time-invariant systems.

Representations of dynamical systems

Systems are often described by equations

$$f(w) = 0, \qquad f : \mathbb{W}^{\mathbb{T}} \to \mathbb{R}^{g},$$

via representations

$$\mathscr{B} = \{\, w \in \mathbb{W}^{\mathbb{T}} \mid f(w) = 0 \,\}. \tag{repr}$$

Note: $f(w) = 0$ is a specific but nonunique description of $\mathscr{B}$.

We will consider systems, which variable space is $\mathbb{R}^{\mathtt{w}}$ and time axis

- $\mathbb{T} = \mathbb{R}$ — continuous-time systems, or
- $\mathbb{T} = \mathbb{Z}$ — discrete-time systems.

Linear time-invariant systems

Properties of a system are defined in terms of its behaviour $\mathscr{B}$

and are translated to equivalent statements in terms of representations.

$$\boxed{\mathscr{B} \text{ is linear if } w, v \in \mathscr{B} \implies \alpha w + \beta v \in \mathscr{B}, \text{ for all } \alpha, \beta \in \mathbb{R}}$$

Recall the shift operator $(\sigma w)(t) = w(t+1)$.

$$\boxed{\mathscr{B} \text{ is time-invariant if } w \in \mathscr{B} \implies \sigma^t w \in \mathscr{B}, \text{ for all } t.}$$

Input/output (I/O) partitioning

Let $\Pi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be a permutation matrix, and define

$$\begin{bmatrix} u \\ y \end{bmatrix} := \Pi w \tag{I/O}$$

(This is just a reordering of the variables.)

The variable u is an input if the behaviour associate with u if free, *i.e.*,

$$\mathscr{B}_u := \{\, u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{T}} \mid \text{there is } y \text{ such that } \Pi^{-1} \begin{bmatrix} u \\ y \end{bmatrix} \in \mathscr{B} \,\} = (\mathbb{R}^{\mathtt{m}})^{\mathbb{T}}.$$

(I/O) is an I/O partitioning for $\mathscr{B}$ if u is free and $\dim(u)$ is maximal.

We will consider systems with given I/O partition and w.l.g. assume that

$$\Pi = I.$$

Difference equations

The difference equation

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_\ell w(t+\ell) = 0, \qquad \text{for all } t \in \mathbb{Z}$$

is more compactly written using the shift operator σ as

$$R_0 \sigma^0 w + R_1 \sigma^1 w + \cdots + R_\ell \sigma^\ell w = 0. \tag{$*$}$$

Define the polynomial matrix

$$R(z) = R_0 + R_1 z + \cdots + R_\ell z^\ell \in \mathbb{R}^{g \times \mathtt{w}}[z]$$

and note that

$$R(\sigma) w = 0$$

is a convenient short hand notation for $(*)$.

Differential equations

The differential equation

$$R_0 \frac{\mathrm{d}^0}{\mathrm{d}t^0} w + R_1 \frac{\mathrm{d}^1}{\mathrm{d}t^1} w + \cdots + R_\ell \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} w = 0$$

is more compactly written as

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0,$$

where again R is the polynomial matrix

$$R(z) = R_0 + R_1 z + \cdots + R_\ell z^\ell.$$

For continuous-time systems, redefine σ as the derivative operator $\mathrm{d}/\mathrm{d}t$, so $R(\sigma)w = 0$ is a difference/differential eqn., depending on the context.

Input/output representation

The difference (in discrete-time) or differential (in continuous-time) eqn

$$P(\sigma)y = Q(\sigma)u, \qquad P \in \mathbb{R}^{g\times \mathrm{p}}[z],\ Q \in \mathbb{R}^{g\times \mathrm{m}}[z] \qquad \text{(I/O eqn)}$$

defines an LTI system $\mathscr{B}$ via

$$\mathscr{B}_{\mathrm{i/o}}(P,Q) := \{\, w = (u,y) \in (\mathbb{R}^{\mathrm{w}})^{\mathbb{N}} \mid \text{(I/O eqn) holds} \,\} \qquad \text{(I/O repr)}$$

If $g = \mathrm{p}$ and $\det(P) \neq 0$, (I/O repr) is called an input/output repr.

The class of system that admit (I/O repr) is called finite dimensional.

Transfer function

Consider a system $\mathscr{B}_{\mathrm{i/o}}(P, Q)$ and let $\mathscr{L}$ be the Laplace transform.

$$P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u \quad \Longrightarrow \quad P(s)Y(s) = Q(s)U(s)$$

where $Y := \mathscr{L}(y)$ and $U := \mathscr{L}(u)$.

The rational function

$$Y(s)U^{-1}(s) = P^{-1}(s)Q(s) =: H(s)$$

is called transfer function.

In the SISO case

$$\frac{Y(s)}{U(s)} = \frac{Q(s)}{P(s)} =: h(s).$$

State of the system

Given
- a system $\mathscr{B}$,
- a "past" trajectory of $\mathscr{B}$, $(\ldots w_{\mathrm{p}}(-2), w_{\mathrm{p}}(-1))$, and
- a "future" input $u_{\mathrm{f}} = (u_{\mathrm{f}}(0), u_{\mathrm{f}}(1), \ldots)$

find the future output y_{f} of $\mathscr{B}$, such that

$$w := (\ldots, w_{\mathrm{p}}(-2), w_{\mathrm{p}}(-1), w_{\mathrm{f}}(0), w_{\mathrm{f}}(1), \ldots)$$

is a trajectory of $\mathscr{B}$.

It turns out that for $\mathscr{B} = \mathscr{B}_{\mathrm{i/o}}(p, q)$, it isn't necessary to know the whole (infinite) past w_{p} in order to find y_{f}!

Suffices to know a finite dimensional, so called "state", vector $x(0)$ of $\mathscr{B}$.

Input/state/output (I/S/O) representation

A finite dimensional LTI system $\mathscr{B} \in \mathscr{L}^{\mathrm{w}}$ admits a representation

$$\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D) := \{\, w := \mathrm{col}(u,y) \in (\mathbb{R}^{\mathrm{w}})^{\mathbb{N}} \mid \exists\, x \in (\mathbb{R}^{\mathrm{n}})^{\mathbb{N}},$$
$$\text{such that } \sigma x = Ax + Bu,\ y = Cx + Du \,\}. \qquad \text{(I/S/O repr)}$$

- x — an auxiliary variable called state
- $\mathrm{n} := \dim(x)$ — state dimension, $\quad \mathbb{R}^{\mathrm{n}}$ — state space
- $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, $B \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$, $C \in \mathbb{R}^{\mathrm{p}\times\mathrm{n}}$, $D \in \mathbb{R}^{\mathrm{p}\times\mathrm{m}}$ — parameters of $\mathscr{B}$
- $\mathrm{m} := \dim(u)$ — input dimension, $\quad \mathrm{p} := \dim(y)$ — output dimension

single input single output (SISO) systems — $\dim(u) = \dim(y) = 1$

multi input multi output (MIMO) systems — $\dim(u) \geq 1$, $\dim(y) \geq 1$

- A — state transition matrix, B — input matrix
- C — output matrix, D — feedthrough matrix
- $\sigma x = Ax + Bu$ — state equation
- $y = Cx + Du$ — output equation

- A shows how $x(t+1)$ depends on $x(t)$ (state transition)
- B shows how $u(t)$ influences $x(t+1)$
- C shows how $y(t)$ depends on $x(t)$
- D shows how $u(t)$ influences $y(t)$ (static I/O relation)

Trivial extension: A, B, C, D functions of t leads to time-varying system

Comparison between I/O and I/S/O representations

- (I/S/O repr) is first order in x and zeroth order in w
- (I/O repr) has no auxiliary variable and is for higher order in w

If the system is single output,

- (I/S/O repr) is vector difference/differential equation
- (I/O repr) is a scalar difference/differential equation

We will consider the problems of constructing I/S/O repr from an I/O one and vice verse, *i.e.*,

$$(P, Q) \mapsto (A, B, C, D) \qquad \text{and} \qquad (A, B, C, D) \mapsto (P, Q)$$

Nonuniqueness of an I/S/O representation

There are two sources of nonuniqueness of (I/S/O repr):

1. redundant states — $\mathtt{n} := \dim(x)$ bigger than "necessary"
2. nonuniqueness of A, B, C, D — choice of state space basis

minimal I/S/O representations — $\dim(x)$ is as small as possible

For any nonsingular matrix $T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and

$$\widetilde{A} = T^{-1}AT, \quad \widetilde{B} = T^{-1}B, \quad \widetilde{C} = CT, \quad \widetilde{D} = D$$

we have that

$$\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D) = \mathscr{B}_{\mathrm{i/s/o}}(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}).$$

Change of state space basis

Consider an LTI system $\mathscr{B} = \mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D)$.

For any $(u, y) \in \mathscr{B}$, there is x, such that

$$\sigma x = Ax + Bu, \qquad y = Cx + Du. \tag{$**$}$$

Let $\widetilde{x} = T^{-1}x$, where $T \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is nonsingular, so that $x = T\widetilde{x}$.

Substituting in $(**)$ and multiplying the first equation by T, we obtain

$$\sigma\widetilde{x} = \underbrace{T^{-1}AT}_{\widetilde{A}}\widetilde{x} + \underbrace{T^{-1}B}_{\widetilde{B}}u, \qquad y = \underbrace{CT}_{\widetilde{C}}\widetilde{x} + \underbrace{D}_{\widetilde{D}}u.$$

$x = T\widetilde{x}$, with T nonsingular, means change of basis in $\mathbb{R}^n$ (from I to T).

Nonuniqueness of an I/O representation

There are two sources of nonuniqueness of (I/O repr):

1. redundant equations — $g := \text{row dim}(P)$ bigger than "necessary"
2. nonuniquencess of P, Q — equivalence of equations

minimal I/O representations — $\text{row dim}(P)$ is as small as possible

In the single output case, P, Q are unique up do a scaling factor, *i.e.*,

$$\widetilde{P} = \alpha P, \qquad \widetilde{Q} = \alpha Q, \qquad \text{for } \alpha \in \mathbb{R}$$

we have that

$$\mathscr{B}_{\text{i/o}}(P, Q) = \mathscr{B}_{\text{i/o}}(\widetilde{P}, \widetilde{Q}).$$

For multi output systems the nonuniqueness of P, Q is more essential.

I/S/O $\mapsto$ transfer function

The transfer function corresponding to a system $\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D)$ is

$$H(s) = C(sI - A)^{-1}B + D.$$

With $X := \mathscr{L}(x)$, $Y := \mathscr{L}(y)$, $U := \mathscr{L}(u)$, we have

$$\begin{aligned} \sigma x &= Ax + Bu \quad &\Longrightarrow& \quad sX = AX + BU \\ y &= Cx + Du \quad &\Longrightarrow& \quad Y = CX + DU \end{aligned}$$

The first equation implies

$$(sI - A)X = BU \quad \Longrightarrow \quad X = (sI - A)^{-1}BU.$$

Substitute in the second equation to get

$$Y = C(sI - A)^{-1}BU + DU = \big(\underbrace{C(sI - A)^{-1}B + D}_{H(s)}\big)U$$