Subspace identification with constraints on the impulse response

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Aims of this talk

identification from a "few" data points

regularize the problem by prior knowledge

use practically meaningful prior knowledge

Motivation: "small data" is also challenging and relevant problem

T	$[0, T_{\min})$	$[T_{\min}, 2 \sim 5T_{\min})$	$(5T_{\min}, \infty)$
problem	ill-posed	ill-conditioned	well-conditioned
cost fun.	—	local minima	$\rightarrow$ convex
solution	non-unique	sensitive	robust
analysis	—	few results	many results

T — number of data points

surprising often we are dealing with "small" T

We are in the "small data" case when doing

nonparameteric identification

parameters $\approx$ # samples $\Longrightarrow$ ill-posed problem

nonlinear identification

large number of parameters $\Longrightarrow$ large $T_{\min}$

fault detection

needs fast real-time identification $\Longrightarrow$ small T

Solution by prior knowledge on parameters

estimation problem: $\min_\theta \underbrace{e(u,y,\theta)}_{\text{model error}}$

regularized problem: $\min_\theta e(u,y,\theta) + \underbrace{\gamma p(\theta)}_{\text{regularizer}}$

$(e.g., p(\theta) = \|\theta_0 - \theta\|)$

alternative problem: $\min_{\theta \in \Theta} e(u,y,\theta)$

$(e.g., \Theta := \{\, \theta \mid \|\theta_0 - \theta\| \leq \rho \,\})$

θ may not be unique!

Prior knowledge on the model behavior

examples of prior:

DC gain $= 1$
rise time < 5 sec
overshoot $< 10\%$

prior is often specified on model trajectories

$\mathscr{B}|_L$ — set of all L-samples long trajectories

Two-stage identification method

1. $(u,y) \mapsto h$ — impulse response estimation

2. $h \mapsto \mathscr{B}$ — realization of h

Plan

Impulse response estimation

Adding constraints

Numerical examples

Model behavior $=$ image of Hankel matrix

$$\mathscr{H}_L(w) := \begin{bmatrix} w(1) & w(2) & w(3) & \cdots & w(T-L+1) \\ w(2) & w(3) & w(4) & \cdots & w(T-L+2) \\ w(3) & w(4) & w(5) & \cdots & w(T-L+3) \\ \vdots & \vdots & \vdots & & \vdots \\ w(L) & w(L+1) & w(L+2) & \cdots & w(T) \end{bmatrix}$$

for exact data $w_\mathrm{d} = (u_\mathrm{d}, y_\mathrm{d})$ and persistently exciting u_d

$$\text{image}\left(\mathscr{H}_L(w_\mathrm{d})\right) = \mathscr{B}|_L$$

therefore, for any $w \in \mathscr{B}|_L$, there is g, such that

$$\mathscr{H}_L(w_\mathrm{d})g = w$$

Computing impulse response is linear prob.

impulse response

$$W = \Big(\underbrace{0,\dots,0}_{\text{zero ini. cond.}}, \begin{bmatrix} I \\ h(0) \end{bmatrix}, \begin{bmatrix} 0 \\ h(1) \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ h(t) \end{bmatrix} \Big)$$

define

$$\mathscr{H}_{\ell+t}(u_{\mathrm{d}}) =: \begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix} \quad \text{and} \quad \mathscr{H}_{\ell+t}(y_{\mathrm{d}}) =: \begin{bmatrix} Y_{\mathrm{p}} \\ Y_{\mathrm{f}} \end{bmatrix}$$

we have

$$\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix} g = \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{zero ini. conditions} \\ \leftarrow \text{impulse input} \end{matrix}$$

$$Y_{\mathrm{f}} \, g = h$$

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Prior $Eh = f \rightsquigarrow$ analytical solution

adding the constraint

$$Eh = f$$

to

$$\underbrace{\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix}}_{\mathscr{A}} g = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix}}_{v}$$

leads to

$$h = Y_{\mathrm{f}} \left((EY_{\mathrm{f}})^{+} f + N(\mathscr{A} N)^{+} \left(v - \mathscr{A} (EY_{\mathrm{f}})^{+} f \right) \right)$$

Prior $E'h \leq f' \rightsquigarrow$ quadratic program

$$\begin{array}{ll} \text{minimize} \quad \text{over } g & \|\mathscr{A} g - v\| \\ \text{subject to} & EY_{\mathrm{f}}g = f \quad \text{and} \quad E' Y_{\mathrm{f}} g \leq f' \end{array}$$

convex problem

solved by active-set methods

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Simulation setup

mass-spring-damper system

$$m\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}y + d\frac{\mathrm{d}}{\mathrm{d}\,t}y + ky = u$$

errors-in-variables setup

$$w_\mathrm{d} = \bar{w} + \widetilde{w}, \qquad \bar{w} \in \bar{\mathscr{B}} \quad \text{and} \quad \widetilde{w} \sim \mathrm{N}(0, s^2 I)$$

$T = 50$ samples

Methods compared

`uy2ss_pk` — two-stage method with prior knowledge

`uy2ss` — two-stage method without prior knowledge

`n4sid` — N4SID method (with default parameters)

Validation criteria

$$e_{\mathscr{B}} = \frac{1}{N}\sum_{k=1}^{N} \frac{\|\bar{\mathscr{B}} - \widehat{\mathscr{B}}^{(k)}\|}{\|\bar{\mathscr{B}}\|} \quad \text{— model error}$$

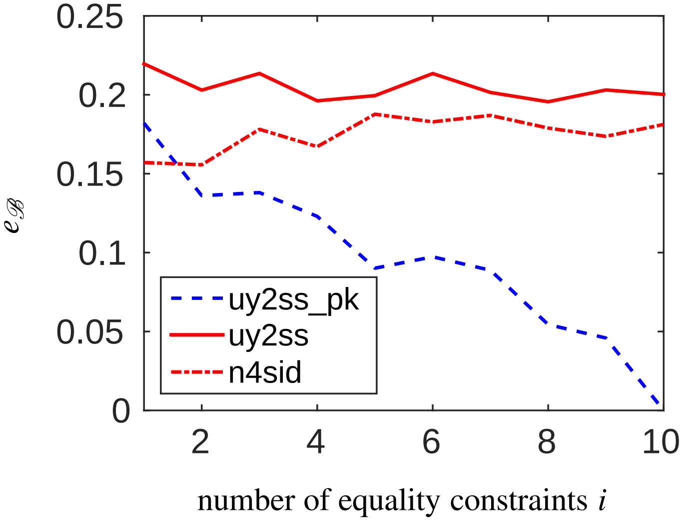
$$e_h = \frac{1}{N}\sum_{k=1}^{N} \frac{\|\bar{h} - \widehat{h}^{(k)}\|}{\|\bar{h}\|} \quad \text{— impulse response error}$$

A single equality constraint

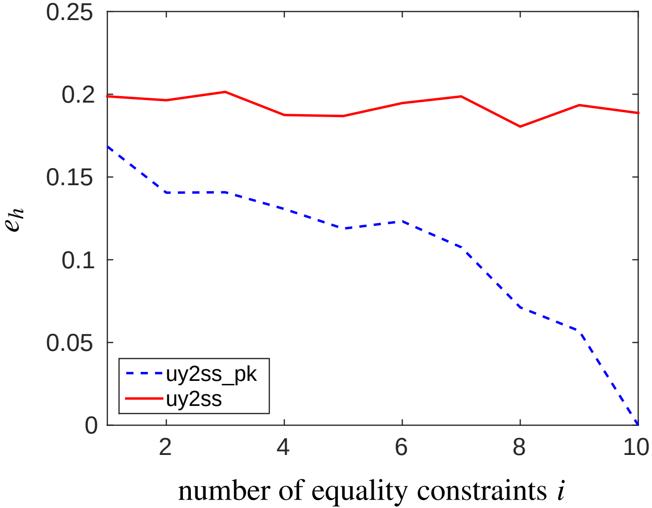
$$E = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \qquad f = E\bar{h}$$

	uh2ss_pk	uh2ss	n4sid
$e_{\mathscr{B}}$	0.1925	0.2218	0.1597
e_h	0.1828	0.2137	—

Multiple equality constraints



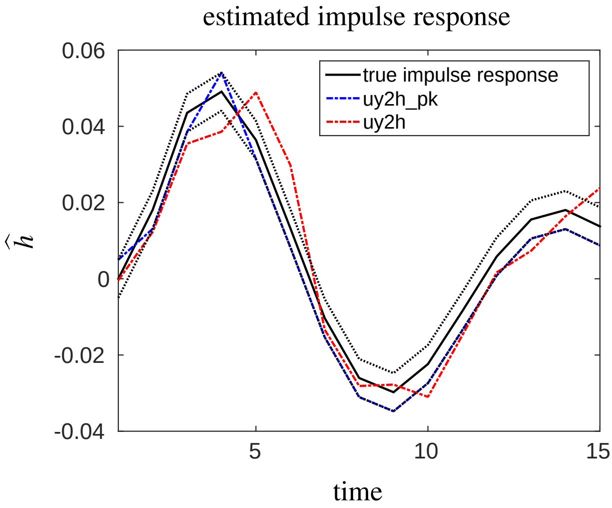
Multiple equality constraints



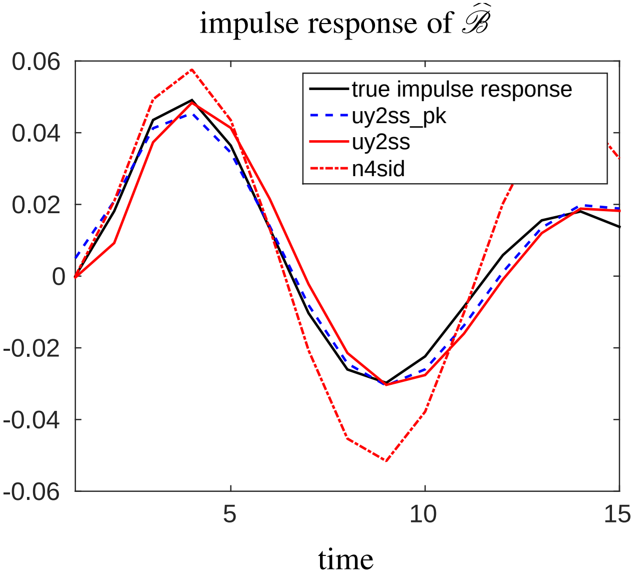
Inequality constraints

	`uh2ss_pk`	`uh2ss`	`n4sid`
$e_{\mathscr{B}}$	0.2291	0.2611	1.4984
e_h	0.2022	0.3235	—

Inequality constraints



Inequality constraints



Conclusions

prior knowledge makes the problem well conditioned

classical approach: Bayesian prior on the parameters

alternative approach: constraints on the model behavior