Identification of autonomous Wiener systems

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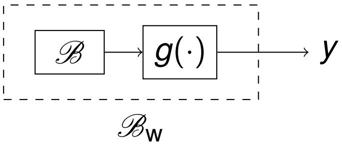
Consider the familiar Wiener system, however, without an input signal



$\mathscr{B}$ — autonomous linear time-invariant subsystem

g — static nonlinear subsystem

$\mathscr{B}_{\mathrm{w}}$ — autonomous Wiener system

The response y is due to initial conditions

existing methods assume zero initial conditions

main result: $\mathscr{B}_{\mathrm{w}} \subseteq$ LTI system (of high order)

this result suggest an identification method

Parameterization of the model

order-n linear subsystem

$$\mathscr{B} = \mathscr{B}(\lambda) := \left\{ z \in \mathbb{R}^{\mathbb{N}} \mid z = \sum_{i=1}^{n} \alpha_i \exp_{\lambda_i},\ \alpha \in \mathbb{C}^n \right\} \qquad (1)$$

degree-d static nonlinear subsystem

$$y = g(z) := \theta^\top v(z), \quad \text{where} \quad v(z) = \begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^d \end{bmatrix} \qquad (2)$$

autonomous Wiener system

$$\mathscr{B}(\lambda, \theta) := \{ y \in \mathbb{R}^{\mathbb{N}} \mid (1,2) \text{ hold for } \alpha \in \mathbb{C}^n \}$$

Main result: $\mathscr{B}(\lambda, \theta)$ is included in an autonomous linear time-invariant system

there is $\lambda_{\rm w}$, such that

$$\mathscr{B}(\lambda, \theta) \subseteq \mathscr{B}(\lambda_{\rm w})$$

the order of the embedding system $\mathscr{B}(\lambda_{\rm w})$ is

$$n_{\rm w} = \binom{n+d}{d} = \frac{(n+1)(n+2)\cdots(n+d)}{d!}$$

its eigenvalues $\lambda_{\rm w}$ are products of d elements of $1 \cup \lambda$

$$\lambda_{{\rm w},i} = \prod_{j=1}^{d} \lambda_{k_{i,j}}, \quad \text{where}, \quad \lambda_0 := 1, \quad k_{i,j} \in \{0, 1, \dots, n\}$$

Strategy: compare the outputs of $\mathscr{B}(\lambda_{\mathrm{w}})$ and $\mathscr{B}(\lambda,\theta)$

the output of $\mathscr{B}(\lambda_{\mathrm{w}})$ is sum-of-damped-exponentials

$$y = \beta_1 \exp_{\lambda_{\mathrm{w},1}} + \cdots + \beta_{n_{\mathrm{w}}} \exp_{\lambda_{\mathrm{w},n_{\mathrm{w}}}}, \quad \beta \in \mathbb{R}^{n_{\mathrm{w}}}$$

consider a general basis element

$$v_j\big(z(t)\big) = \big(z(t)\big)^j = \left(\sum_{i=1}^{n} \alpha_i \lambda_i^t\right)^j$$

v_j is a sum-of-damped-exponentials

$$v_j\big(z(t)\big) = \sum_{i=1}^{n_j} \gamma_i \mu_{i,j}^t, \quad \text{where } \mu_{i,j}^t = \prod_{\ell=1}^{j} \lambda_{k_{i,j,\ell}}$$

then, the output

$$y(t) = g\big(z(t)\big) = \theta v\big(z(t)\big)$$

is also a sum-of-damped-exponentials

$$y(t) = \sum_{i=1}^{n_\mathrm{w}} \zeta_i \lambda_{\mathrm{w},i}^t, \quad \text{where } \lambda_\mathrm{w} = \bigcup_{j=0}^{d} \bigcup_{i=0}^{j} \mu_{i,j}$$

λ_w = all products of d elements of $1 \cup \lambda(\mathscr{B})$

however, $\zeta \in$ subset of $\mathbb{R}^{n_\mathrm{w}} \quad \Longrightarrow \quad \mathscr{B}(\lambda, \theta) \subseteq \mathscr{B}(\lambda_\mathrm{w})$

Corollary: link between λ_w and λ

the symmetric, rank-1, d-way tensor

$$T := \lambda \times_1 \lambda \times_2 \cdots \times_{d-1} \lambda$$

has as unique elements $\lambda_{w,1}, \ldots, \lambda_{w,n_w}$

Identification problem

given: monomial basis v and a finite trajectory

$$y_{\mathrm{d}} = \big(y_{\mathrm{d}}(1), \dots, y_{\mathrm{d}}(T)\big)$$

of an autonomous Wiener system $\mathscr{B}(\lambda, \theta)$

find: the order n and parameters $\widehat{\lambda}, \widehat{\theta}$, such that

$$\mathscr{B}(\lambda, \theta) = \mathscr{B}(\widehat{\lambda}, \widehat{\theta})$$

Procedure for identification of autonomous Wiener system

1. identify $\mathscr{B}_{\text{w}}$ from the given output data
2. compute the linear subsystem $\mathscr{B}$ from $\mathscr{B}_{\text{w}}$
3. compute the nonlinear subsystem g from $\mathscr{B}_{\text{w}}$ and $\mathscr{B}$

1) identification of $\mathscr{B}_{\mathrm{w}}$ from y

minimal number of samples needed: $T_{\mathrm{min}} = 2n_{\mathrm{w}} + 1$

can be collected from n_{w} experiments with $n_{\mathrm{w}} + 1$ samples

issue: $\mathscr{B}_{\mathrm{w}}$ is a stiff system

2) computation of $\mathscr{B}$ from $\mathscr{B}_\mathrm{w}$

rank-1 factorization of symmetric, *d*-way tensor

$$T\big(\lambda(\mathscr{B}_\mathrm{w})\big) = \lambda \times_1 \lambda \times_2 \cdots \times_{d-1} \lambda$$

issue: order of the eigenvalues $\lambda(\mathscr{B}_\mathrm{w})$

the combinatorial number of factorizations can be avoided

3) computation of g from $\mathscr{B}_{\mathrm{w}}$ and $\mathscr{B}$

simultaneous rank-1 factorization of d tensors

this is a structured data fusion problem

if g has first order term, there is a simple solution