# Identification of autonomous Wiener systems

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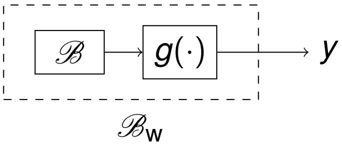
# Consider the familiar Wiener system, however, without an input signal




$\mathscr{B}$ — autonomus linear time-invariant subsystem

$g$ — static nonlinear subsystem

$\mathscr{B}_{\mathrm{w}}$ — autonomous Wiener system

# The response $y$ is due to initial conditions

existing methods assume zero initial conditions

main result: $\mathscr{B}_{\mathrm{w}} \subseteq$ LTI system (of high order)

this result suggest an identification method

# Parameterization of the model

order-$n$ linear subsystem

$$\mathscr{B} = \mathscr{B}(\lambda) := \left\{ z \in \mathbb{R}^{\mathbb{N}} \mid z = \sum_{i=1}^{n} \alpha_i \exp_{\lambda_i},\ \alpha \in \mathbb{C}^n \right\} \qquad (1)$$

degree-$d$ static nonlinear subsystem

$$y = g(z) := \theta^\top v(z), \quad \text{where} \quad v(z) = \begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^d \end{bmatrix} \qquad (2)$$

autonomous Wiener system

$$\mathscr{B}(\lambda, \theta) := \{\, y \in \mathbb{R}^{\mathbb{N}} \mid \text{(1,2) hold for } \alpha \in \mathbb{C}^n \}$$

# Main result: $\mathscr{B}(\lambda, \theta)$ is included in an autonomous linear time-invariant system

there is $\lambda_{\mathrm{w}}$, such that

$$\mathscr{B}(\lambda, \theta) \subseteq \mathscr{B}(\lambda_{\mathrm{w}})$$

the order of the equivalent system $\mathscr{B}(\lambda_{\mathrm{w}})$ is

$$n_{\mathrm{w}} = \binom{n+d}{d} = \frac{(n+1)(n+2)\cdots(n+d)}{d!}$$

its eigenvalues $\lambda_{\mathrm{w}}$ are products of $d$ elements of $1 \cup \lambda$

$$\lambda_{\mathrm{w},i} = \prod_{j=1}^{d} \lambda_{k_{i,j}}, \quad \text{where,} \quad \lambda_0 := 1, \quad k_{i,j} \in \{0, 1, \dots, n\}$$

# Strategy: compare the outputs of $\mathscr{B}(\lambda_\mathrm{w})$ and $\mathscr{B}(\lambda, \theta)$

the output of $\mathscr{B}(\lambda_\mathrm{w})$ is sum-of-damped-exponentials

$$y = \alpha_1 \exp_{\lambda_{\mathrm{w},1}} + \cdots + \alpha_{n_\mathrm{w}} \exp_{\lambda_{\mathrm{w},n_\mathrm{w}}}, \quad \alpha \in \mathbb{R}^{n_\mathrm{w}}$$

consider a general basis element

$$v_j\big(y(t)\big) = \big(y(t)\big)^j = \left(\sum_{i=1}^{n} \alpha_i \lambda_i^t\right)^j$$

$v_j$ is a sum-of-damped-exponentials

$$v_j\big(z(t)\big) = \sum_{i=1}^{n_j} \beta_i \mu_{i,j}^t, \quad \text{where } \mu_{i,j}^t = \prod_{\ell=1}^{j} \lambda_{k_{i,j,\ell}}$$

then, the output

$$y(t) = g\big(z(t)\big) = \theta v\big(z(t)\big)$$

is also a sum-of-damped-exponentials

$$y(t) = \sum_{i=1}^{n_\mathrm{w}} \gamma_i \lambda_{\mathrm{w},i}^t, \quad \text{where } \lambda_\mathrm{w} = \bigcup_{j=0}^{d} \bigcup_{i=0}^{j} \mu_{i,j}$$

$\lambda_\mathrm{w} =$ all products of $d$ elements of $1 \cup \lambda(\mathscr{B})$

however, $\gamma \in$ subset of $\mathbb{R}^{n_\mathrm{w}} \quad \Longrightarrow \quad \mathscr{B}(\lambda, \theta) \subseteq \mathscr{B}(\lambda_\mathrm{w})$

# Corollary: link between $\lambda_\mathrm{w}$ and $\lambda$

the symmetric, rank-1, $d$-way tensor

$$T := \lambda \times_1 \lambda \times_2 \cdots \times_{d-1} \lambda$$

has as unique elements $\lambda_{\mathrm{w},1}, \ldots, \lambda_{\mathrm{w},n_\mathrm{w}}$

# Identification problem

given: monomial basis $v$ and a finite trajectory

$$y_{\mathrm{d}} = \big(y_{\mathrm{d}}(1), \dots, y_{\mathrm{d}}(T)\big)$$

of an autonomous Wiener system $\mathscr{B}(\lambda, \theta)$

find: the order $n$ and parameters $\widehat{\lambda}, \widehat{\theta}$, such that

$$\mathscr{B}(\lambda, \theta) = \mathscr{B}(\widehat{\lambda}, \widehat{\theta})$$

# Procedure for identification of autonomous Wiener system

1. identify $\mathscr{B}_{\mathrm{w}}$ from the given output data
2. compute the linear subsystem $\mathscr{B}$ from $\mathscr{B}_{\mathrm{w}}$
3. compute the nonlinear subsystem $g$ from $\mathscr{B}_{\mathrm{w}}$ and $\mathscr{B}$

# 1) identification of $\mathscr{B}_{\mathrm{w}}$ from $y$

minimal number of samples needed: $T_{\min} = 2n_{\mathrm{w}} + 1$

can be collected from $n_{\mathrm{w}}$ experiments with $n_{\mathrm{w}} + 1$ samples

issue: $\mathscr{B}_{\mathrm{w}}$ is a stiff system

# 2) computation of $\mathscr{B}$ from $\mathscr{B}_\mathrm{w}$

rank-1 factorization of symmetric, $d$-way tensor

$$T\big(\lambda(\mathscr{B}_\mathrm{w})\big) = \lambda \times_1 \lambda \times_2 \cdots \times_{d-1} \lambda$$

issue: order of the eigenvalues $\lambda(\mathscr{B}_\mathrm{w})$

requires combinatorial number of factorizations

# 3) computation of $g$ from $\mathscr{B}_{\mathrm{w}}$ and $\mathscr{B}$

simultaneous rank-1 factorization of $d$ tensors

this is a structured data fusion problem

if $g$ has first order term, there is a simple solution