Applications of common factor computation in signal processing

Ivan Markovsky, Antonio Fazzi, and Nicola Guglielmi

VRIJE UNIVERSITEIT BRUSSEL

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Issue: exact properties are lost under noise

data	a set of polynomials
exact property	existence of a common factor

distance problem:

how far is the data from having the property?

Approximate GCD computation ...

given polynomials $p^1, \dots, p^N$ and a natural number d

$$\begin{array}{ll} \text{minimize} & \text{over } \widehat{p}^1, \dots, \widehat{p}^N \quad \sum_{i=1}^{N} \|p^i - \widehat{p}^i\|_2^2 \\ \text{subject to} & \deg\big(\gcd(\widehat{p}^1, \dots, \widehat{p}^N)\big) \geq d \end{array}$$

$\gcd(\widehat{p}^1, \dots, \widehat{p}^N)$ — greatest common divisor of $\widehat{p}^1, \dots, \widehat{p}^N$

. . . has applications in signal processing

Blind FIR system identification

Distance to uncontrollability

Common dynamics estimation

Problem formulation (exact data)

given: output observations $y^1, \dots, y^N$ of FIR system

(generated by unknown inputs $u^1, \dots, u^N$)

find: the impulse response h of the FIR system

Result: it is a GCD problem

if

- at least $N = 2$ responses $y^1, \ldots, y^N$ are given
- $u^1, \ldots, u^N$ have finite support, and
- $\gcd\big(u^1(z), \ldots, u^N(z)\big) = 1$

then

$$h(z) = \alpha \gcd\big(y^1(z), \ldots, y^N(z)\big)$$

α — unknown scaling factor

Problem formulation (noisy data)

assumption:

$$y_{\rm d}^i = \bar{y}^i + \widetilde{y}^i, \quad \text{for } i = 1, \dots, N$$

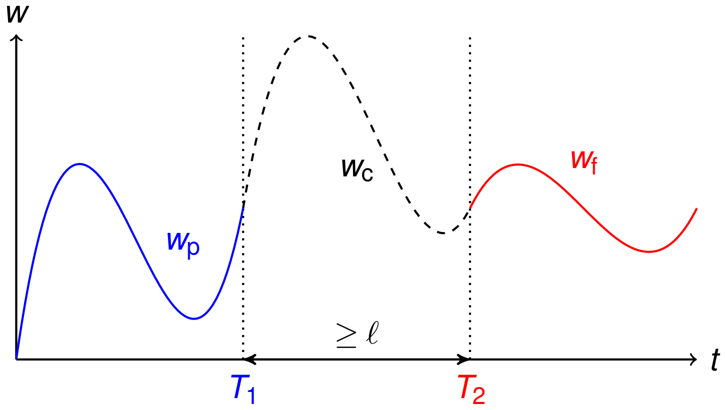
$\widetilde{y}$ — zero mean, white, Gaussian noise

maximum-likelihood estimator:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{N} \|y_{\rm d}^i - \widehat{y}^i\|_2^2 \\ \text{subject to} & \widehat{y}^i = \widehat{h} \star \widehat{u}^i, \quad \text{for } i = 1, \dots, N \end{array}$$

$\rightsquigarrow$ approximate common factor computation

Controllability property, defined in terms of the system's trajectories $w = \begin{bmatrix} u \\ y \end{bmatrix}$



for all w_p, $w_f \in \mathscr{B}$, there is w_c, such that $w_p \wedge w_c \wedge w_f \in \mathscr{B}$

("$\wedge$" denotes "concatenation" of trajectories)

Linear time-invariant systems

polynomial representation (σ — unit shift)

$$\mathscr{B}_{\mathrm{i/o}}(p,q) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \mid p(\sigma)y = q(\sigma)u \right\}$$

fact: $\mathscr{B}_{\mathrm{i/o}}(p,q)$ is controllable iff (p,q) are co-prime

distance between systems

$$\operatorname{dist}\big(\mathscr{B}_{\mathrm{i/o}}(p,q), \mathscr{B}_{\mathrm{i/o}}(\widehat{p},\widehat{q})\big) := \left\| \begin{bmatrix} q \\ p \end{bmatrix} - \begin{bmatrix} \widehat{q} \\ \widehat{p} \end{bmatrix} \right\|_2$$

Distance to uncontrollability problem

given $\mathscr{B}_{\text{i/o}}(p,q)$, find the uncontrollability radius

$$\text{dist}_{\text{unctr}}(\mathscr{B}) := \min_{\widehat{\mathscr{B}} \in \overline{\mathscr{L}_{\text{ctrb}}}} \text{dist}(\mathscr{B}, \widehat{\mathscr{B}})$$

$\overline{\mathscr{L}_{\text{ctrb}}}$ — set of uncontrollable LTI systems

$\rightsquigarrow$ approximate common factor computation with $d = 1$

Model-based common dynamics estimation

given systems $\mathscr{B}_1, \ldots, \mathscr{B}_N$, find their "common dynamics"

$$\mathscr{B} := \mathscr{B}_1 \cap \cdots \cap \mathscr{B}_N$$

kernel representation $\mathscr{B}_i = \{ y \mid R^i(\sigma) y = 0 \}$

fact

$$R(z) = \text{gcd}\big(R^1(z), \ldots, R^N(z)\big)$$

Data-driven common dynamics estimation

given data $y_1 \in \mathscr{B}_1, \ldots, y_N \in \mathscr{B}_N$

find the common dynamics $\mathscr{B} := \mathscr{B}_1 \cap \cdots \cap \mathscr{B}_N$

model-based approach: $(y_1, \ldots, y_N) \mapsto (R^1, \ldots, R^N) \mapsto \mathscr{B}$

data-driven approach: $(y_1, \ldots, y_N) \mapsto \mathscr{B}$

Subspace common dynamics estimation

assume that $\text{rank}\big(\text{hankel}(y^i)\big)$ is maximal

then $\mathscr{B}_i = \text{span}\big(\text{hankel}(y^i)\big)$

finding $\mathscr{B}$ is a common subspace computation problem