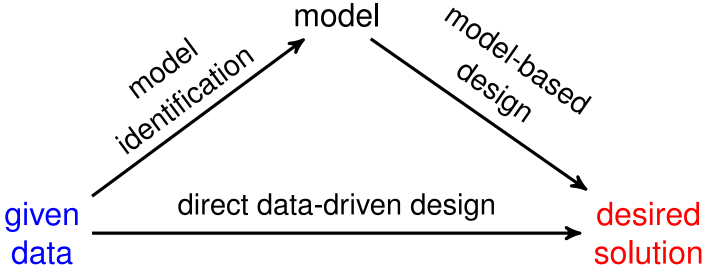
System theory without transfer functions and state-space? Yes, it's possible!

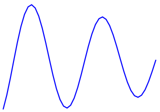
Ivan Markovsky

The current trend in systems & control is bypassing parametric model identification



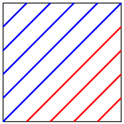
Example: data-driven forecasting of sum-of-damped-exponentials signal

given: $\underbrace{y(1),\ldots,y(t)}_{\text{"past" data}}$ find: $\underbrace{y(t+1),\ldots,y(2t)}_{\text{"future" samples}}$



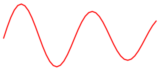
given data

$\mapsto$ rank



$\leq r$ $\mapsto$

matrix completion problem



prediction

$\rightsquigarrow$ Hankel structured low-rank completion

The interpolation/approximation problem considered is missing data estimation

given

1) trajectory w_{d} of LTI system $\mathscr{B}$, and
2) partially specified trajectory w of $\mathscr{B}$

complete w if a completion exists, otherwise

approximate and complete w

Plan

Data-driven representation of LTI systems

Interpolation of trajectories

Special cases and generalizations

Outline

Data-driven representation of LTI systems

Interpolation of trajectories

Special cases and generalizations

A dynamical system $\mathscr{B}$ is a set of signals

$$w \in \mathscr{B} \quad \leftrightarrow \quad \text{"}w \text{ is trajectory of } \mathscr{B}\text{"}$$
$$\leftrightarrow \quad \text{"}\mathscr{B} \text{ is exact model for } w\text{"}$$

$\mathscr{B}$ is linear system $\quad :\Longleftrightarrow \quad$ $\mathscr{B}$ is subspace

$\mathscr{B}$ is time-invariant $\quad :\Longleftrightarrow \quad$ $\sigma\mathscr{B} = \mathscr{B}$

$(\sigma w)(t) := w(t+1)$ — shift operator

$\sigma\mathscr{B} := \{\sigma w \mid w \in \mathscr{B}\}$

The set of linear time-invariant systems $\mathscr{L}$ has structure characterized by set of integers

the dimension of $\mathscr{B} \in \mathscr{L}$ is determined by

$\mathbf{m}(\mathscr{B})$ — number of inputs

$\mathbf{n}(\mathscr{B})$ — order ($=$ minimal state dimension)

$\boldsymbol{\ell}(\mathscr{B})$ — lag ($=$ observability index)

J.C. Willems, From time series to linear systems.
Part I, Finite dimensional linear time invariant systems, Automatica, 22(561–580), 1986

Identifiability: $w_\mathrm{d} \in \mathscr{B}$ specifies $\mathscr{B} \in \mathscr{L}$

define $\quad \widehat{\mathscr{B}} := \operatorname{span}\{\, w_\mathrm{d}, \sigma w_\mathrm{d}, \sigma^2 w_\mathrm{d}, \dots \}$

$\widehat{\mathscr{B}} \in \mathscr{L}$ and $\widehat{\mathscr{B}} \subseteq \mathscr{B}$

identifiability condition: $\quad \widehat{\mathscr{B}} = \mathscr{B}$

J.C. Willems, From time series to linear systems.
Part II, Exact modelling, Automatica, 22(675–694), 1986

We aim to obtain finite horizon results

restriction of w and $\mathscr{B}$ to finite interval $[1, L]$

$$w|_L := \big(w(1), \ldots, w(L)\big), \quad \mathscr{B}|_L := \{\, w|_L \mid w \in \mathscr{B} \,\}$$

$$\dim \mathscr{B}|_L = \mathbf{m}(\mathscr{B}) L + \mathbf{n}(\mathscr{B}), \quad \text{for all } L \geq \boldsymbol{\ell}(\mathscr{B})$$

for $\mathscr{B}, \mathscr{B}' \in \mathscr{L}$, $\quad \mathscr{B} = \mathscr{B}' \quad$ if and only if

$$\mathscr{B}|_L = \mathscr{B}'|_L, \text{ for } L = \max\{\, \boldsymbol{\ell}(\mathscr{B}), \boldsymbol{\ell}(\mathscr{B}') \,\} + 1$$

Shifting and cutting w_{d} leads to Hankel matrix

for $w_{\mathrm{d}} = \big(w_{\mathrm{d}}(1), \dots, w_{\mathrm{d}}(T)\big)$ and $1 \leq L \leq T$

$$\mathscr{H}_L(w_{\mathrm{d}}) := \Big[(\sigma^0 w_{\mathrm{d}})|_L \quad (\sigma^1 w_{\mathrm{d}})|_L \quad \cdots \quad (\sigma^{T-L} w_{\mathrm{d}})|_L \Big]$$

define $\quad \widehat{\mathscr{B}}_L := \operatorname{image} \mathscr{H}_L(w_{\mathrm{d}})$

then, $\quad \widehat{\mathscr{B}}_L \subseteq \mathscr{B}|_L$

Identifiability condition that is verifiable from $w_\mathrm{d} \in \mathscr{B}|_T$ and $\big(\mathbf{m}(\mathscr{B}), \boldsymbol{\ell}(\mathscr{B}), \mathbf{n}(\mathscr{B})\big)$

$$
\begin{aligned}
\widehat{\mathscr{B}} = \mathscr{B} \quad &\iff \quad \widehat{\mathscr{B}}|_{\boldsymbol{\ell}(\mathscr{B})+1} = \mathscr{B}|_{\boldsymbol{\ell}(\mathscr{B})+1} \\
&\iff \quad \dim \widehat{\mathscr{B}}|_{\boldsymbol{\ell}(\mathscr{B})+1} = \dim \mathscr{B}|_{\boldsymbol{\ell}(\mathscr{B})+1}
\end{aligned}
$$

$\mathscr{B}$ is identifiable from $w_\mathrm{d} \in \mathscr{B}|_T$ if and only if

$$
\operatorname{rank} \mathscr{H}_{\boldsymbol{\ell}(\mathscr{B})+1}(w_\mathrm{d}) = \big(\boldsymbol{\ell}(\mathscr{B}) + 1\big)\mathbf{m}(\mathscr{B}) + \mathbf{n}(\mathscr{B})
$$

Nonparametric repr. $\mathscr{B}|_L = \text{image}\, \mathscr{H}_L(w_\text{d})$

$\widehat{\mathscr{B}}_L \subseteq \mathscr{B}|_L$, $L \geq \boldsymbol{\ell}(\mathscr{B})$, equality holds if and only if

$$\text{rank}\, \mathscr{H}_L(w_\text{d}) = L\mathbf{m}(\mathscr{B}) + \mathbf{n}(\mathscr{B})$$

sufficient conditions ("fundamental lemma"):

1. $w_\text{d} = \begin{bmatrix} u_\text{d} \\ y_\text{d} \end{bmatrix}$
2. $\mathscr{B}$ controllable
3. $\mathscr{H}_{L+\mathbf{n}(\mathscr{B})}(u_\text{d})$ full row rank

J.C. Willems et al., A note on persistency of excitation
Systems & Control Letters, (54)325–329, 2005

Outline

Data-driven representation of LTI systems

Interpolation of trajectories

Special cases and generalizations

Data-driven interpolation is missing data recovery

given:

$$
\begin{array}{lcl}
w_{\mathrm{d}} \in \mathscr{B}|_T & — & \text{“data” trajectory} \\
w|_{I_{\mathrm{given}}} & — & \text{partially specified trajectory}
\end{array}
$$

($w|_{I_{\mathrm{given}}}$ selects the elements of w, specified by I_{given})

find:

$$\widehat{w} \in \mathscr{B}|_L, \quad \text{such that} \quad \widehat{w}|_{I_{\mathrm{given}}} = w|_{I_{\mathrm{given}}}$$

Solution may not exist

A1: $\operatorname{rank} \mathscr{H}_L(w_{\mathrm{d}}) = \mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B})$

A2: $w|_{I_{\text{given}}}$ has exact completion $w \in \mathscr{B}|_L$

$$\operatorname{rank} \begin{bmatrix} \mathscr{H}_L(w_{\mathrm{d}})|_{I_{\text{given}}} & w|_{I_{\text{given}}} \end{bmatrix} = \operatorname{rank} \mathscr{H}_L(w_{\mathrm{d}})|_{I_{\text{given}}}$$

($M|_{I_{\text{given}}}$ selects the submatrix of M with rows in I_{given})

A1 $+$ A2 $\quad \Longrightarrow \quad$ exact solution exists

Solution may not be unique

when recovered solution is exact, *i.e.*, $\widehat{w} = w$?

A3: there are "enough" given samples I_{given}

$$\operatorname{rank} \mathscr{H}_L(w_{\text{d}})|_{I_{\text{given}}} = \operatorname{rank} \mathscr{H}_L(w_{\text{d}})$$

A2 and A3 are verifiable from the data
A1 requires in addition $\mathbf{m}(\mathscr{B}), \boldsymbol{\ell}(\mathscr{B}), \mathbf{n}(\mathscr{B})$

Data-driven interpolation method: solve system of linear equations

there is g, such that $w = \mathscr{H}_L(w_\mathrm{d})g$

method:

1. solve $\quad w|_{I_\mathrm{given}} = \mathscr{H}_L(w_\mathrm{d})|_{I_\mathrm{given}} g$
2. define $\quad \widehat{w} := \mathscr{H}_L(w_\mathrm{d})g$

Outline

Special cases

simulation

- given data: initial conditions and input
- to-be-found: output (exact interpolation)

smoothing

- given data: noisy trajectory
- to-be-found: ℓ_2-optimal approximation

tracking

- given data: to-be-tracked trajectory
- to-be-found: ℓ_2-optimal approximation

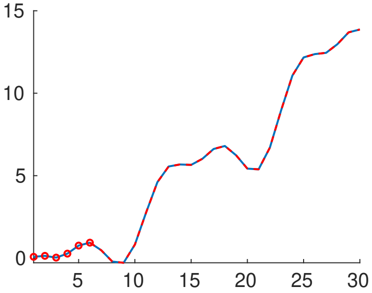
Simulation is special case of interpolation

```
%% interpolation points
t_given = 1:n;
w_given = rand(n, 1);

%% data-driven interpolation
g = H(t_given, :) \ w_given;
w = H * g;

%% check the result
ws = initial(B, w_given, L-n);
norm(ws - w(n:L)) % = 0?

%% results
plot(w), hold on,
plot(n:L, ws, 'r--')
plot(t_given, w_given, 'rO')
```



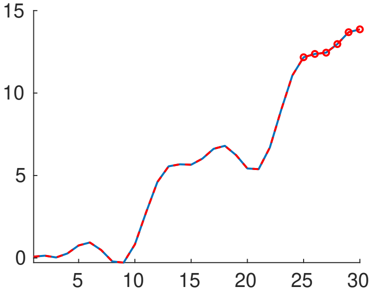
Simulation with “terminal conditions”

```
%% interpolation points
t_given = L-n+1:L;
w_given = w(t_given);

%% data-driven interpolation
g = H(t_given, :) \ w_given;
w_final = H * g;

%% check the result
norm(w - w_final) % = 0?

%% results
plot(w_final), hold on,
plot(w, 'r--')
plot(t_given, w_given, 'rO')
```

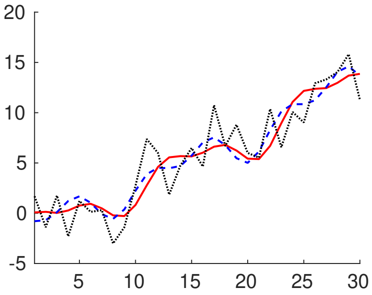


Example: data-driven approximation (errors-in-variables Kalman smoothing)

```
%% noisy trajectory
w0 = w; % exact trajectory
w  = w0 + 2 * randn(L, 1);

%% data-driven approximation
wh = H * pinv(H) * w;

%% results
plot(w0, 'r-'), hold on
plot(wh, 'b--')
plot(w,  ':k')
```



Generalizations

w_d not exact / noisy

maximum-likelihood estimation
$\rightsquigarrow$ Hankel structured low-rank approximation/completion
parametric, non-convex optimization problem

nuclear norm and ℓ_1-norm relaxations
$\rightsquigarrow$ nonparametric, convex optimization problems

nonlinear systems

results for special classes of nonlinear systems:
Volterra, Wiener-Hammerstein, bilinear, ...

efficient / real-time computation

non-parametric version of the Kalman filter

ℓ_1-norm regularization

in the noise-free case g can be chosen sparse

$$\|g\|_0 \leq L\mathbf{m}(\mathscr{B}) + \mathbf{n}(\mathscr{B})$$

impose sparsity in the case of noisy data

$$\text{minimize} \quad \text{over } g \quad \|w|_{I_{\text{given}}} - \mathscr{H}_L(w_{\text{d}})|_{I_{\text{given}}} g\| + \lambda \|g\|_1$$

hyper-parameter: $\quad \lambda \quad \leftrightarrow \quad \mathbf{m}(\mathscr{B}), \mathbf{n}(\mathscr{B})$

System theory without transfer function and state space representations is possible

data-driven representation

leads to general, simple, practical methods

interpolation/approximation of trajectories

simulation, filtering and control are special cases
assumes only LTI dynamics; no hyper parameters

future work

efficient / real-time algorithms
sensitivity / statistical analysis
nonlinear systems