Constructive proof of the fundamental lemma

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The fundamental lemma gives data-driven finite horizon representation of LTI system $\mathscr{B}$

$$\mathscr{B}|_L = \text{image}\, \mathscr{H}_L(w_\text{d}) \qquad \text{(DD-REPR)}$$

assumptions:

A0 $w_\text{d} = \begin{bmatrix} u_\text{d} \\ y_\text{d} \end{bmatrix}$ is a trajectory of an LTI system $\mathscr{B}$

A1 $\mathscr{B}$ is controllable

A2 u_d is persistently exciting of order $L + n$

Decoding the notation $\mathscr{B}|_L = \text{image}\, \mathscr{H}_L(w_d)$

$\mathscr{B}$ — system's behavior, *i.e.*, set of trajectories

$\mathscr{B}|_L$ — restriction of $\mathscr{B}$ to the interval $[1, L]$

$w_d := \big(w_d(1), \dots, w_d(T_d)\big)$ — "data" trajectory

$$\mathscr{H}_L(w_d) := \begin{bmatrix} w_d(1) & w_d(2) & \cdots & w_d(T_d - L + 1) \\ \vdots & \vdots & & \vdots \\ w_d(L) & w_d(L+1) & \cdots & w_d(T_d) \end{bmatrix}$$

$\text{PE}(u_d) := \max L$, such that $\mathscr{H}_L(u_d)$ is f.r.r.

This talk addresses the following questions

proof by contradiction

What is the meaning/interpretation of the conditions?

sufficiency of the conditions

How conservative are they? Can they be improved?

conjecture

The extra PE of order n is generically not needed.
What are the nongeneric cases when it is needed?

Our answers / main results are

constrictive proof in the single-input case

$$\mathrm{PE}(u_\mathrm{d}) = n_u \iff u_\mathrm{d} \in \mathscr{B}_u|_{T_\mathrm{d}}, \text{ where } \mathscr{B}_u \text{ is autonomous LTI of order } n_u$$

shows that the FL is nonconservative

conjecture: it is conservative in the multi-input case

characterizes the nongeneric cases

they correspond to special initial conditions

Necessary and sufficient condition for the data-driven representation

$$\operatorname{rank} \mathscr{H}_L(w_\mathrm{d}) = mL + n, \qquad \text{(GPE)}$$

nonconservative (necessary and sufficient)

general no I/O partitioning and controllability

verifiable from w_d with prior knowledge of (m, n)

I. Markovsky and F. Dörfler, Identifiability in the Behavioral Setting, 2020

The fundamental lemma is input design result

input design problem

choose u_d, so that (DD-REPR) holds for any initial cond.

refined problem statement

find nonconservative conditions on $u_{\rm d}$ and $\mathscr{B}$, under which

$$\text{for } \forall\, w_{\rm d,ini},\; w_{\rm d,ini} \wedge w_{\rm d} \in \mathscr{B}|_{T_{\rm ini}+T_{\rm d}} \text{ satisfies (GPE)} \quad \text{(GOAL)}$$

subproblem: find $w_{\rm ini}$ that minimize rank $\mathscr{H}_L(w_{\rm d})$

Obvious necessary conditions

A0: exact representation requires exact data and input design requires input/output partition

A1: for uncontrollable $\mathscr{B} = \mathscr{B}_{\text{ctr}} \oplus \mathscr{B}_{\text{aut}}$

- $w_{\text{d}} \in \mathscr{B} \implies w_{\text{d}} = w_{\text{d,ctr}} + w_{\text{d,aut}},\ w_{\text{d,ctr}} \in \mathscr{B}_{\text{ctr}},\ w_{\text{d,aut}} \in \mathscr{B}_{\text{aut}}$
- $w_{\text{d,aut}}$ is completely determined by $w_{\text{d,ini}}$
- there is $w_{\text{d,ini}}$, such that $w_{\text{d,aut}} = 0 \implies$ (GPE) doesn't hold

A2′: u_{d} is persistently exciting of order L

- since u is an input, $\Pi_u \mathscr{B}|_L = \mathbb{R}^{\mathbf{m}(\mathscr{B})L}$
- for (GPE) to hold true, image $\mathscr{H}_L(u_{\text{d}}) = \mathbb{R}^{\mathbf{m}(\mathscr{B})L}$
- equivalently, $\mathscr{H}_L(u_{\text{d}})$ must be full row-rank

Find the minimal k, such that (GOAL) holds under A0, A1, and $\mathrm{PE}(u_\mathrm{d}) = L + k$

first, we solve the subproblem

find w_{ini}^* that minimize rank $\mathscr{H}_L(w_d)$

then, we check (GPE) for w_ini^*

$\rightsquigarrow$ minimal $k \implies$ nonconservative PE condition

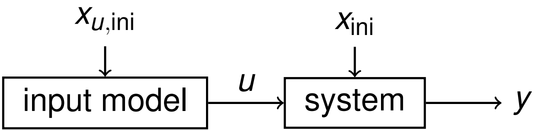
The PE condition is equivalent to existence of an LTI input model

$$u_{\mathrm{d}} \in (\mathbb{R})^{T_{\mathrm{d}}} \quad \text{and} \quad \mathrm{PE}(u_{\mathrm{d}}) = n_u$$

$$\Updownarrow$$

$$u_{\mathrm{d}} \in \mathscr{B}_u|_{T_{\mathrm{d}}} \text{ — autonomous LTI}, \quad T_{\mathrm{d}} \geq 2n_u - 1$$

$$\mathscr{B}_u = \mathscr{B}_{\mathrm{ss}}(A_u, C_u) \text{ with } \big(A_u, x_{u,\mathrm{ini}}\big) \text{ controllable}$$



Augmented system with the input model

$\mathscr{B}_{\text{ext}} = \mathscr{B}_{\text{ss}}(A_{\text{ext}}, C_{\text{ext}})$, with $x_{\text{ext}} = \left[\begin{smallmatrix} x_u \\ x \end{smallmatrix}\right]$

$$A_{\text{ext}} = \begin{bmatrix} A_u & 0 \\ BC_u & A \end{bmatrix} \quad C_{\text{ext}} = \begin{bmatrix} C_u & 0 \\ DC_u & C \end{bmatrix}$$

$\mathscr{B}_{\text{ext}} = \mathscr{B}_{\text{ss}}\left(A'_{\text{ext}}, C'_{\text{ext}}\right)$, where $x'_{\text{ext}} = \left[\begin{smallmatrix} x_u \\ Vx_u + x \end{smallmatrix}\right]$

$$A'_{\text{ext}} = \begin{bmatrix} A_u & 0 \\ 0 & A \end{bmatrix}, \quad C'_{\text{ext}} = \begin{bmatrix} C_u & 0 \\ C' & C \end{bmatrix}, \quad C' := DC_u - CV$$

V is solution of the Sylvester equation $AV - VA_u = BC_u$

The nongeneric cases correspond to special initial conditions $x_{\text{ini}} = -Vx_{u,\text{ini}}$

which eliminates from w_{d} the transient due to $\mathscr{B}$

then, $\operatorname{rank} \mathscr{H}_L(w_{\text{d}}) \leq \text{PE}(u_{\text{d}}) = n_u$

next, we show that $\operatorname{rank} \mathscr{H}_L(w_{\text{d}}) = n_u$

assume simple eigenvalues $\lambda_{u,1}, \dots, \lambda_{u,n_u}$ of $\mathscr{B}_u$

$$u_{\mathrm{d}} = \sum_{i=1}^{n_u} a_i \exp_{\lambda_{u,i}}$$

assume simple eigenvalues $\lambda_1, \dots, \lambda_n$ of $\mathscr{B}$

$$y_{\mathrm{d}} = \sum_{i=1}^{n_u} b_i \exp_{\lambda_{u,i}} + \underbrace{\sum_{j=1}^{n} c_j \exp_{\lambda_j}}_{\text{transient}}$$

- $b_i = H(e^{\mathbf{i}\lambda_{u,i}}) a_i$, where $H(z) := C(Iz - A)^{-1} B + D$
- $w_{\mathrm{ini}} = w_{\mathrm{ini}}^* \implies c_j = 0$

using Vandermonde matrix, we rewrite $(u_\mathrm{d}, y_\mathrm{d})$

$$u_\mathrm{d} = \underbrace{\begin{bmatrix} \lambda_{u,1}^1 & \cdots & \lambda_{u,n_u}^1 \\ \vdots & & \vdots \\ \lambda_{u,1}^T & \cdots & \lambda_{u,n_u}^T \end{bmatrix}}_{V_T(\lambda_u)} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_{n_u} \end{bmatrix}}_{a} = V_T(\lambda_u) a$$

and

$$y_\mathrm{d} = V_T(\lambda_u) \underbrace{\begin{bmatrix} H(e^{\mathbf{i}\lambda_{u,1}}) & & \\ & \ddots & \\ & & H(e^{\mathbf{i}\lambda_{u,n_u}}) \end{bmatrix}}_{H(\lambda_u)} \begin{bmatrix} a_1 \\ \vdots \\ a_{n_u} \end{bmatrix}$$

$$= V_T(\lambda_u) \underbrace{H(\lambda_u) a}_{b} = V_T(\lambda_u) b$$

then, for w_d, we obtain

$$w_\mathrm{d} = \Pi_T \begin{bmatrix} V_T(\lambda_u) \\ V_T(\lambda_u) H(\lambda_u) \end{bmatrix} a$$

$\Pi_T \in \mathbb{R}^{2T \times 2T}$ permutation, such that $w_\mathrm{d} = \Pi_T \left[\begin{smallmatrix} u_\mathrm{d} \\ y_\mathrm{d} \end{smallmatrix}\right]$

finally, the Hankel matrix is expressed as

$$\mathscr{H}_L(w_\mathrm{d}) = \underbrace{\Pi_L \begin{bmatrix} V_L(\lambda_u) \\ V_L(\lambda_u) H(\lambda_u) \end{bmatrix}}_{W_L} \underbrace{\begin{bmatrix} a & \Lambda_u a & \Lambda_u^2 a & \cdots & \Lambda_u^{T-L} a \end{bmatrix}}_{\text{controllability matrix of } (\Lambda_u, a)}$$

$$\Lambda_u := \mathrm{diag}(\lambda_{u,1}, \ldots, \lambda_{u,n_u})$$

(Λ_u, a) is controllable because $\mathrm{PE}(u_\mathrm{d}) = n_u$

1. $a_i \neq 0$ for all i
2. $\lambda_{u,i} \neq \lambda_{u,j}$ for all $i \neq j$

for $k \leq n$, W_L is full column rank

- with $W_L = \begin{bmatrix} w^1 & \dots & w^{n_u} \end{bmatrix}$, w^i are trajectories ($w^i \in \mathscr{B}|_L$)
- $\lambda_{u,i} \neq \lambda_{u,j}$ for all $i \neq j \quad \Longrightarrow \quad$ independent responses

$$\operatorname{rank} \mathscr{H}_L(w_\mathrm{d}) = \begin{cases} L + k, & \text{for } k = 1, \dots, n \\ L + n, & \text{for } k = n+1, \dots \end{cases}$$

$k = n$ is the minimal value for (GPE) to hold

Comments

the zeros of $\mathscr{B}$ don't play role in the analysis

simple eigenvalues assumptions can be relaxed

"robustifying" the conditions

exact condition:	robust version:
$a_i \neq 0$, for all i	$a_i > \varepsilon$
$\lambda_{u,i} \neq \lambda_{u,j}$, for all $i \neq j$	the $\lambda_{u,i}$'s are "well spread"

conjecture: in multi-input case, A2 can be tightened, $\text{PE}(u_\text{d}) = n +$ controllability index $\mathscr{B}$