Closed-loop data-driven simulation

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Notation

$\mathtt{w}$ — # of external variables, $\mathtt{m}$ — # of inputs, $\mathtt{p}$ — # of outputs

$\mathbb{N} := \{1, 2, \dots\}$ — time axis

$\mathscr{B}(A, B, C, D)$ — the system defined by $\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$

Restriction of the system behavior $\mathscr{B}$ to the interval $1, 2, \dots, t$

$$\mathscr{B}_t := \{ w_\mathrm{p} \in (\mathbb{R}^\mathtt{w})^t \mid \text{there is } w_\mathrm{f} \text{ such that } (w_\mathrm{p}, w_\mathrm{f}) \in \mathscr{B} \}$$

$\mathrm{lag}(\mathscr{B})$ — lag of $\mathscr{B}$ (observability index of I/S/O repr.)

$\mathrm{order}(\mathscr{B})$ — order of $\mathscr{B}$

We assume that an input/output partition of the variables is given.

The simulation problem

Classical simulation problem: Given

- system $\mathscr{B} := \mathscr{B}(A, B, C, D)$,
- input $u \in (\mathbb{R}^\mathtt{m})^t$, and
- initial conditions $x_\mathrm{ini} \in \mathbb{R}^\mathtt{n}$,

find the response y of $\mathscr{B}$ to u and ini. cond. x_ini.

Data-driven simulation problem: Given

- trajectory $w_\mathrm{d} = (u_\mathrm{d}, y_\mathrm{d}) \in (\mathbb{R}^\mathtt{w})^T$ of $\mathscr{B}$,
- input $u \in (\mathbb{R}^\mathtt{m})^t$, and
- initial trajectory $w_\mathrm{ini} \in (\mathbb{R}^\mathtt{w})^{T_\mathrm{ini}}$, $w_\mathrm{ini} \in \mathscr{B}_{T_\mathrm{ini}}$,

find the response y of $\mathscr{B}$ to u, such that $\big(w_\mathrm{ini}, (u, y)\big) \in \mathscr{B}_{T_\mathrm{ini}+t}$.

Notes:

- $\mathscr{B}$ is specified implicitly by w_d,
- the initial condition x_ini is specified implicitly by w_ini.

Algorithm 1: data-driven simulation, using I/S/O repr.

1. identification $w_\mathrm{d} \mapsto (A, B, C, D)$
2. observer $\big(w_\mathrm{ini}, (A, B, C, D)\big) \mapsto x_\mathrm{ini}$
3. classical simulation $\big(u, x_\mathrm{ini}, (A, B, C, D)\big) \mapsto y$

Can we find y without deriving an explicit representation of $\mathscr{B}$?

Notation for Hankel matrices

Given a signal $w = (w(1), \dots, w(T))$ and $t \leq T$, define

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & w(3) & \cdots & w(T-t+1) \\ w(2) & w(3) & w(4) & \cdots & w(T-t+2) \\ w(3) & w(4) & w(5) & \cdots & w(T-t+3) \\ \vdots & \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & w(t+2) & \cdots & w(T) \end{bmatrix}$$

block-Hankel matrix with t block-rows, composed of w

Basic idea

Assuming that w_d is a trajectory of $\mathscr{B}$ (exact data),

lin. comb. of the columns of $\mathscr{H}_t(w_d)$ are trajectories of $\mathscr{B}$, *i.e.*,

$$\text{for all } g, \quad \mathscr{H}_t(w_d)g \in \mathscr{B}_t$$

Under additional conditions—persistency of excitation of u_d and controllability of $\mathscr{B}$—every trajectory can be generated that way.

In what follows, we assume that these conditions are satisfied.

$\Longrightarrow$ computing the response of $\mathscr{B}$ to given input and initial conditions from data w_d, requires choosing a suitable g

Construction of responses from data

Problem: Find y, such that $(w_{ini}, (u, y)) \in \mathscr{B}$, where w_{ini}, u are given, and $\mathscr{B}$ is implicitly defined by w_d.

There is g, such that

$$\mathscr{H}_{T_{ini}+t}(w_d)g = (w_{ini}, (u, y)).$$

The eqns with RHS y, define y, for given g. The others restrict g.

Generic data-driven simulation algorithm:

1. compute any solution g of the equations with RHS w_{ini}, u
2. substitute g in the equations for y

Define

$$U := \mathscr{H}_{T_{ini}+t}(u_d), \qquad Y := \mathscr{H}_{T_{ini}+t}(y_d)$$

and the partitionings

$$U =: \begin{bmatrix} U_p \\ U_f \end{bmatrix}, \qquad Y =: \begin{bmatrix} Y_p \\ Y_f \end{bmatrix}.$$

Algorithm 2: data-driven simulation

1. compute the least norm solution of

$$\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix} g = \begin{bmatrix} u_{ini} \\ y_{ini} \\ u \end{bmatrix}.$$

2. compute $y := Y_f g$.

Special case $u = 0$: free response

Allows to compute an observability matrix $\mathscr{O}$ of $\mathscr{B}$ from data, by finding $n \geq \text{order}(\mathscr{B})$ linearly indep. free responses.

Let $\ell_{\max}$ be an upper bound for the lag of $\mathscr{B}$ and take $T_{\text{ini}} = \ell_{\max}$.

Algorithm 3: compute an observability matrix $\mathscr{O}$

1. compute the least norm solution of

$$\begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \end{bmatrix} G = \begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ 0 \end{bmatrix}$$

2. compute $Y := Y_{\text{f}} G$
3. compute a rank revealing factorization $Y = \mathscr{O} X_{\text{ini}}$

Special case $w_{\text{ini}} = 0$: zero initial cond. response

Let h be the impulse response of $\mathscr{B}$, and define

$$\mathscr{T}_t(h) := \begin{bmatrix} h(0) & & & & \\ h(1) & h(0) & & & \\ h(2) & h(1) & h(0) & & \\ \vdots & \ddots & \ddots & \ddots & \\ h(t-1) & \cdots & \cdots & h(1) & h(0) \end{bmatrix}$$

For any $w = \text{col}(u, y) \in \mathscr{B}_t$,

$$y = \mathscr{O} x_{\text{ini}} + \mathscr{T}_t(h) u$$

We can compute a basis for $\mathscr{B}_{0,t} := \text{image}\left(\mathscr{T}_t(h)\right)$ from data, by finding $t\mathtt{m}$ lin. indep. zero initial cond. responses.

Algorithm 4: compute a basis of $\mathscr{B}_{0,t}$

1. compute the least norm solution of

$$\begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \end{bmatrix} G = \begin{bmatrix} 0 \\ 0 \\ \mathscr{H}_{t,t\mathtt{m}}(u_{\text{d}}) \end{bmatrix}$$

2. compute $Y_0 := Y_{\text{f}} G$

Then $\text{image}(Y_0) = \text{image}\left(\mathscr{T}_t(h)\right) = \mathscr{B}_{0,t}$.

Special case $w_{\text{ini}} = 0$, $u = I\delta$: impulse response

With the same construction we can find the first t Markov parameters of $\mathscr{B}$, which is a system identification method.

Algorithm 5: compute the impulse response

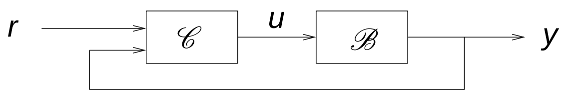
1. compute the least norm solution of

$$\begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \end{bmatrix} G = \begin{bmatrix} 0 \\ 0 \\ \text{col}(I_{\mathtt{m}}, 0) \end{bmatrix}$$

2. compute $h := Y_{\text{f}} G$

Closed-loop system $\mathscr{B}_{\mathscr{C}}$

The interconnected system plant-controller



is

$$\mathscr{B}_{\mathscr{C}} = \mathscr{B}_{\text{ext}} \cap \mathscr{C}$$

where

$$\mathscr{B}_{\text{ext}} := \{ (r, w) \in (\mathbb{R}^{\mathtt{r}+\mathtt{w}})^{\mathbb{T}} \mid w \in \mathscr{B} \}$$

Note: Open-loop is a special case when $\mathscr{C}$ does not restrict y

Closed-loop data-driven simulation

Given

- traj. $w_{\text{d}} = (u_{\text{d}}, y_{\text{d}}) = \big(w_{\text{d}}(1), \dots, w_{\text{d}}(T)\big)$ of an LTI system $\mathscr{B}$;
- LTI controller $\mathscr{C}$, with inputs r, y and output u; and
- reference signal $r_{\text{r}} = \big(r_{\text{r}}(1), \dots, r_{\text{r}}(T_{\text{r}})\big)$;

find the set of responses w_{r} of $\mathscr{B}_{\mathscr{C}}$ to the reference signal r_{r}.

We aim to compute for given w_{d}, $\mathscr{C}$, and r_{r}, the signals w_{r}, s.t.

$$(r_{\text{r}}, w_{\text{r}}) \in \mathscr{B}_{\mathscr{C}} \iff \begin{cases} w_{\text{r}} \in \mathscr{B} \\ (r_{\text{r}}, w_{\text{r}}) \in \mathscr{C} \end{cases} \iff \begin{cases} \exists\, g, \text{ s.t. } w_{\text{r}} = M(\sigma) g \\ R(\sigma) \operatorname{col}(r_{\text{r}}, w_{\text{r}}) = 0 \end{cases}$$

where (assuming $\mathscr{B}$ is controllable)

$$\mathscr{B} =: \text{image}\big(M(\sigma)\big) \quad \text{and} \quad \mathscr{C} =: \{ (r, w) \mid R(\sigma) \operatorname{col}(r, w) = 0 \}.$$

↑ image representation of $\mathscr{B}$ ↑ kernel representation of $\mathscr{C}$

Assuming that u is persistent,

$$w_{\text{r}} = M(\sigma) g \iff w_{\text{r}} = \mathscr{H}_{T_{\text{r}}}(w_{\text{d}}) g$$

Polynomial operator ↔ banded Toeplitz operator

Banded upper-triang. Toeplitz matrix, parameterized by $r \in \mathbb{R}^{1 \times \mathtt{r}}[z]$

$$\mathscr{T}_t(r) := \begin{bmatrix} r_0 & r_1 & \cdots & r_n & 0 & \cdots & 0 \\ 0 & r_0 & r_1 & \cdots & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & r_0 & r_1 & \cdots & r_n \end{bmatrix}.$$

Association of polynomials of degree $\leq n$ and vectors in $\mathbb{R}^{n+1}$

$$a(z) = a_0 + a_1 z + \cdots + a_n z^n \quad \leftrightarrow \quad \operatorname{col}(a_0, a_1, \dots, a_n)$$

We have

$$a(z) b(z) \quad \leftrightarrow \quad \mathscr{T}_{\deg(b)+1}(a) b = \mathscr{T}_{\deg(a)+1}(b) a$$

Under controllability and persistency of excitation

$$\begin{cases} w_r = M(\sigma)g \\ R(\sigma)\operatorname{col}(r_r, w_r) = 0 \end{cases} \iff \begin{cases} w_r = \mathscr{H}_{T_r}(w_d)g \\ \mathscr{T}_{T_r}(R)\operatorname{col}(r_r, w_r) = 0 \end{cases}$$

Let $R =: \begin{bmatrix} R_r & R_w \end{bmatrix}$, where $R_r \in \mathbb{R}^{1\times r}[z]$ and $R_w \in \mathbb{R}^{1\times w}[z]$.

$$\mathscr{T}_{T_r}(R)\operatorname{col}(r_r, w_r) = 0 \implies \mathscr{T}_{T_r}(R_w)w_r = -\mathscr{T}_{T_r}(R_r)r_r$$

$$\underbrace{\mathscr{T}_{T_r}(R_w)\mathscr{H}_{T_r}(w_d)}_{A} g = \underbrace{-\mathscr{T}_{T_r}(R_r)r_r}_{b}$$

Let g_0 be particular solution and N be basis for $\ker(A)$.

$$\mathscr{G} := \{\, g_0 + Nz \mid z \in \mathbb{R}^{\operatorname{coldim}(N)} \,\}$$

and the set of responses w_r of $\mathscr{B}_{\mathscr{C}}$ to r_r is

$$\mathscr{W}_r = \mathscr{H}_{T_r}(w_d)\mathscr{G} = \{\, \underbrace{\mathscr{H}_{T_r}(w_d)g_0}_{w_{r,0}} + \mathscr{H}_{T_r}(w_d)Nz \mid z \in \mathbb{R}^{\operatorname{coldim}(N)} \,\}$$

Algorithm for closed-loop data-driven simulation

Input: $w_d \in (\mathbb{R}^w)^T$, $R \in \mathbb{R}^{1\times(r+w)}[z]$, and $r_r \in (\mathbb{R}^r)^{T_r}$

1: Compute the least-norm solution g_0 of the system of equations $\mathscr{T}_{T_r}(R_w)\mathscr{H}_{T_r}(w_d)g = -\mathscr{T}_{T_r}(R_r)r_r$

2: Let $w_{r,0} := \mathscr{H}_{T_r}(w_d)g_0$

3: Compute a matrix N which columns form a basis for the column span of $\mathscr{T}_{T_r}(R_w)\mathscr{H}_{T_r}(w_d)$

4: Let N_w be a basis for the column span of $\mathscr{H}_{T_r}(w_d)N$

Output: $w_{r,0}$ and N_w

Example

- $\mathscr{B} = \{\, (u,y) \in (\mathbb{R}^2)^{\mathbb{N}} \mid \sigma y - y = u \,\}$
- $\mathscr{C}_\alpha = \{\, (r,u,y) \mid u = -\alpha(r-y) \,\}$, with $\alpha = 0.5$
- w_d — response of $\mathscr{B}_{\mathscr{C}_{0.1}}$ to random r, $y(0) = 0$; $T = 50$
- $r_r(t) = \begin{cases} 0, & \text{for } t = 1,\dots,5 \\ 1, & \text{for } t = 6,\dots,10. \end{cases}$

Kernel representation of $\mathscr{C}_\alpha$ is given by $R = \begin{bmatrix} \alpha & 1 & -\alpha \end{bmatrix}$.

$\mathscr{B}$ is controllable and u_d is persistently exciting of order 25 $(T_r + \operatorname{order}(\mathscr{B}) = 11)$

Apply the data-driven closed-loop algorithm to get $w_{r,0}$ and N_w.

We should verify that

$$\underbrace{\{\, w_{r,0} + N_w z \mid z \in \mathbb{R}^{\operatorname{coldim}(N_w)} \,\}}_{\widehat{\mathscr{W}_r}} = \underbrace{\{\, y \in (\mathbb{R}^w)^{T_r} \mid (r_r, y) \in \mathscr{B}_{\mathscr{C}_\alpha}|_{T_r} \,\}}_{\mathscr{W}_r}$$

In order to do this, we will show that

1. $w_{r,0}$ is a response of $\mathscr{B}_{\mathscr{C}_\alpha}$ driven by r_r,
2. N_w is a zero input response of $\mathscr{B}_{\mathscr{C}_\alpha}$, and
3. $\dim(\widehat{\mathscr{W}_r}) = \dim(\mathscr{W}_r)$.

Indeed

$$\text{Items 1 and 2} \implies \widehat{\mathscr{W}_r} \subseteq \mathscr{W}_r$$

and item 3 ensures that equality holds.

Note that

$$\dim(\mathscr{W}_\mathrm{r}) = \operatorname{order}(\mathscr{B}) + \operatorname{order}(\mathscr{C}_\alpha) = 1$$

and it turns out that

$$\dim(\widehat{\mathscr{W}_\mathrm{r}}) = \operatorname{coldim}(N_w) = \operatorname{rank}\left(\mathscr{H}_{T_\mathrm{r}}(w_\mathrm{d})N\right) = 1$$

so item 3 holds.

Verifying items 1 and 3, we check that

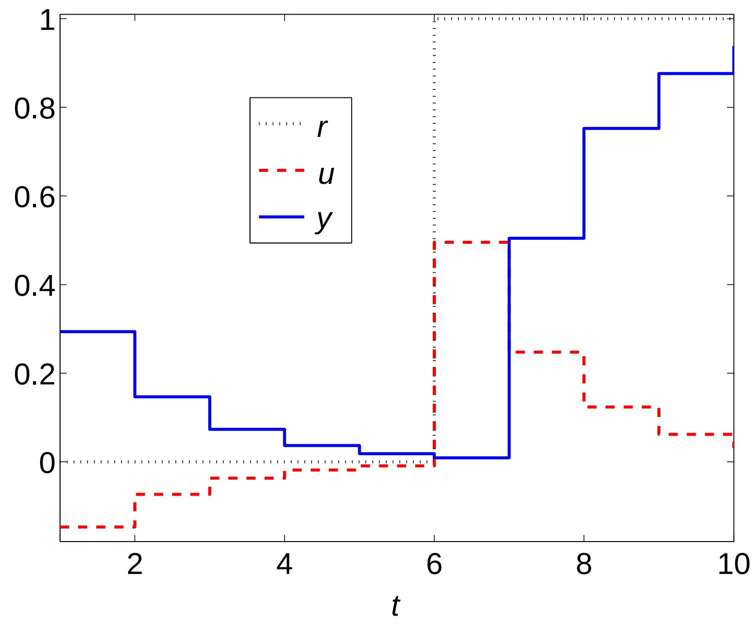
$$(r_\mathrm{r}, w_{\mathrm{r},0}) \in \mathscr{B}_{\mathscr{C}_\alpha} \quad , \quad (0, N_w) \in \mathscr{B}_{\mathscr{C}_\alpha}$$

These are state estimation problems. It turns out that

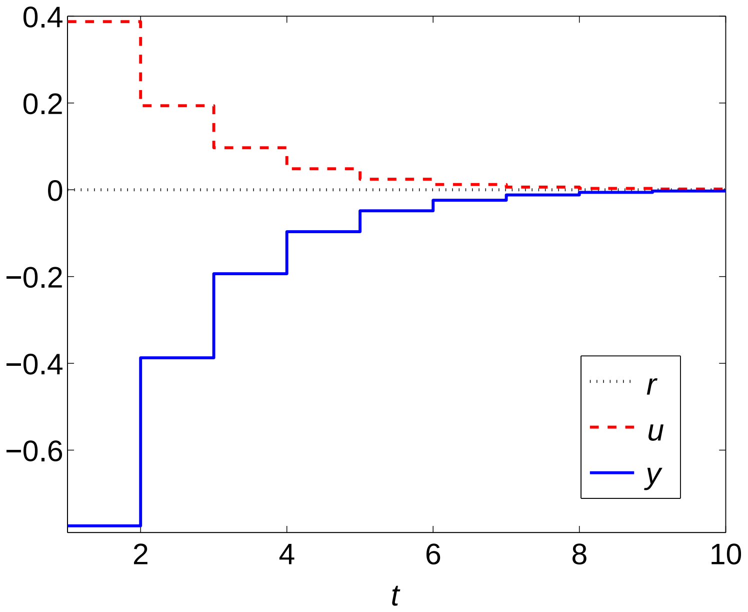
$$y(0) = 0.2937 \rightsquigarrow (r_\mathrm{r}, w_{\mathrm{r},0}) \quad \text{and} \quad y(0) = -0.7746 \rightsquigarrow (0, N_w)$$

so items 1 and 2 hold.

$r = r_\mathrm{r}$ and $w = (u, y) = w_{\mathrm{r},0}$



$r = 0$ and $w = (u, y) = N_w$



Thank you