Data-driven systems theory, signal processing, and control

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Free fall as a dynamical system

setup: mass *m* falling in gravitational field

- $w \in (\mathbb{R}^2)^{\mathbb{R}_+}$ — position in 2D plane
- $v := \dot{w} \in (\mathbb{R}^2)^{\mathbb{R}_+}$ — velocity
- $w(0), v(0) \in \mathbb{R}^2$ — initial condition

task: given initial condition, find the trajectory *w*

first work out the model-based approach

1. using physics, derive model (include friction force $-\gamma\dot{w}$)
2. write a function `w = fall(w0, v0, t, m, gamma)`

Modeling from first principles leads to affine time-invariant state-space model

second law of Newton + the law of gravity

$$m\ddot{w} = m\begin{bmatrix} 0 \\ -9.81 \end{bmatrix} + f, \quad \text{where} \quad w(0) = w_{\text{ini}} \text{ and } \dot{w}(0) = v_{\text{ini}}$$

- 9.81 — gravitational constant
- $f = -\gamma v$ — force due to friction in the air

state $x := (w_1, \dot{w}_1, w_2, \dot{w}_2, x_5)$, where $x_5 = -9.81$

initial state $x_{\text{ini}} := (w_{\text{ini},1}, v_{\text{ini},1}, w_{\text{ini},2}, v_{\text{ini},2}, -9.81)$

Modeling from first principles leads to affine time-invariant state-space model

$$\dot{x} = \begin{bmatrix} 0 & 1 & & & \\ 0 & -\gamma/m & & & \\ & & 0 & 1 & \\ & & 0 & -\gamma/m & 1 \\ & & & & 0 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} w_{\text{ini},1} \\ v_{\text{ini},1} \\ w_{\text{ini},2} \\ v_{\text{ini},2} \\ -9.81 \end{bmatrix}$$

$$w = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x$$

data: N, T_{d}-samples long discretized trajectories

MATLAB function for free fall simulation

```
function [w, t, sys] = fall(w0, v0, t, m, gamma)
a1 = [0 1; 0 -gamma / m];
a  = blkdiag(a1, a1, 0); a(4, 5) = 1;
c = zeros(2, 5); c(1, 1) = 1; c(2, 3) = 1;
sys = ss(a, [], c, []);
x0 = [w0(1); v0(1); w0(2); v0(2); -9.81];
[w, t] = initial(sys, x0, t);
```

the Control Toolbox `initial` simulates the LTI system

Data-driven approach to free fall simulation

simulate $T_\mathrm{d} = 100$-samples trajectories

- $N = 10$ "data" trajectories $w_\mathrm{d}^1, \dots, w_\mathrm{d}^N$ and
- one "to-be-predicted" trajectory w

verify the data "informativity" condition

$$\operatorname{rank} \begin{bmatrix} w_\mathrm{d}^1 & \cdots & w_\mathrm{d}^N \end{bmatrix} = 5$$

implement and verify the data-driven method

1. solve $$\begin{bmatrix} w_\mathrm{d}^1(1) & \cdots & w_\mathrm{d}^N(1) \\ w_\mathrm{d}^1(2) & \cdots & w_\mathrm{d}^N(2) \\ w_\mathrm{d}^1(3) & \cdots & w_\mathrm{d}^N(3) \end{bmatrix} g = \begin{bmatrix} w(1) \\ w(2) \\ w(3) \end{bmatrix}$$
2. define $w := \begin{bmatrix} w_\mathrm{d}^1 & \cdots & w_\mathrm{d}^N \end{bmatrix} g$

Solution

```
%% simulation parameters
m = 1;          % mass
gamma = 0.5; % firction coefficient
N = 20;         % number of experiments
T = 101;        % number of samples
MC = 100;       % Monte-Carlo repetitions

%% simulate data using the function fall.m
t = linspace(0, 1, T); % time vector
Wini = rand(2, N);     % initial positions
Vini = 5 * rand(2, N); % initial velocities
W0 = []; w0 = {};      % collect the trajectories
for i = 1:N
  w0i = fall(Wini(:, i), Vini(:, i), t, m, gamma);
  W0 = [W0 vec(w0i')]; w0{i} = w0i;
end
n = rank(W0) % -> 5
```

Solution

```
%% simulate the to-be-predicted trajectory
wini = rand(2, 1); vini = 5 * rand(2, 1);
w_new = fall(wini, vini, t, m, gamma);

%% validation criterion:
e = @(wh) 100 * norm(w_new - wh, 'fro') ...
                / norm(w_new, 'fro');

%% direct method with pseudo-inverse
g = pinv([Wini; Vini; ones(1, N)]) ...
       * [wini; vini; 1];
wh_dd = reshape(W0 * g, 2, T)';

%% results
e(wh_dd) % -> 0
```

Dynamical systems as set of signals

$\mathscr{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ — q-variate discrete-time system

consider the identification problem $w_d \mapsto \mathscr{B}$

given data $w_d \in \mathscr{B}$, find $\widehat{\mathscr{B}} \subset (\mathbb{R}^q)^{\mathbb{Z}}$, such that $w_d \in \widehat{\mathscr{B}}$

questions

- is there a solution?
- if so, is it unique?
- if so, is $\widehat{\mathscr{B}} = \mathscr{B}$?

Solution

always exist "trivial" solutions: $\widehat{\mathscr{B}} = \{ w_\mathrm{d} \}$ and $\widehat{\mathscr{B}} = (\mathbb{R}^q)^\mathbb{Z}$

the problem find $\widehat{\mathscr{B}} \subset (\mathbb{R}^q)^\mathbb{Z}$, such that $w_\mathrm{d} \in \widehat{\mathscr{B}}$ is ill-posed

additional conditions are needed for well-posedness, *e.g.*,

1. $\widehat{\mathscr{B}} \in \mathscr{M}$ — a given class of systems (*e.g.*, linear systems)
2. find the "simplest" exact model in $\widehat{\mathscr{B}} \in \mathscr{M}$
   $\rightsquigarrow$ $\widehat{\mathscr{B}} = \mathrm{mpum}(w_\mathrm{d})$ — the most powerful unfalsified model

if $w_\mathrm{d} \in \mathscr{B} \in \mathscr{M}$, $\mathrm{mpum}(w_\mathrm{d}) \subseteq \mathscr{B}$

additional conditions are needed for $\mathrm{mpum}(w_\mathrm{d}) = \mathscr{B}$
$\rightsquigarrow$ identifiability conditions

Consider the case of linear static system $\mathscr{B}$

$\mathscr{L}_0^q$ class of linear static systems

- static system with q variables: $\mathscr{B} \subset \mathbb{R}^q$
- linear static system — subspace $\mathscr{B}$ of $\mathbb{R}^q$
- complexity of $\mathscr{B} \in \mathscr{L}_0^q$ — $:= \mathit{dim}\,\mathscr{B}$

identification problem:

- given data: $\mathscr{W}_\mathrm{d} = \{ w_\mathrm{d}^1, \dots, w_\mathrm{d}^N \}$, $w_\mathrm{d}^i \in \mathscr{B} \in \mathscr{L}^q$
- find $\widehat{\mathscr{B}} = \mathrm{mpum}(\mathscr{W}_\mathrm{d})$ in the model class $\mathscr{L}_0^q$
- when is $\widehat{\mathscr{B}} = \mathscr{B}$?

Solution

$$\widehat{\mathscr{B}} = \text{mpum}(w_\text{d}) = \text{span}\,\mathscr{W}_\text{d}$$

$\widehat{\mathscr{B}} \subseteq \mathscr{B}$, equality holds iff $\dim \widehat{\mathscr{B}} = \dim \mathscr{B}$

computational procedure for checking $\widehat{\mathscr{B}} = \mathscr{B}$

$$\text{rank} \begin{bmatrix} w_\text{d}^1 & \cdots & w_\text{d}^N \end{bmatrix} = \dim \mathscr{B}$$

note that prior knowledge of $\dim \mathscr{B}$ is needed

Follow-up questions for self-work

how to representation $\mathscr{B} \in \mathscr{L}_0^q$?

how to find a representation from data $\mathscr{W}_\text{d} \subset \mathscr{B}$?

how to find a representation from “noisy data”

$$w_\text{d} = \overline{w} + \widetilde{w}, \quad \text{where } \overline{w} \in \mathscr{B} \in \mathscr{L}^q \text{ and } \widetilde{w} \text{ is noise}$$

Interpretation of $\dim \mathscr{B}|_L = \mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B})$

question: does it make sense? explain

answer:

$\mathscr{B}\|_L$	subspace of $\mathbb{R}^{qL}$
$\dim \mathscr{B}\|_L$	# of degrees of freedom in choosing $w \in \mathscr{B}\|_L$
$\mathbf{m}(\mathscr{B})$	# of degrees of freedom per time step
$\mathbf{m}(\mathscr{B})L$	# of degrees of freedom due to the inputs
$\mathbf{n}(\mathscr{B})$	# of degrees of freedom due to the initial conditions

Interpretation of $\mathscr{H}_L(w_\mathrm{d})$

what is the system theory meaning of $\mathscr{H}_L(w_\mathrm{d})$?

answer: by definition

$$\mathscr{H}_L(w_\mathrm{d}) := \begin{bmatrix} (\sigma^0 w_\mathrm{d})|_L & (\sigma^1 w_\mathrm{d})|_L & \cdots & (\sigma^{T_\mathrm{d}-L} w_\mathrm{d})|_L \end{bmatrix}$$

$\Longrightarrow$ every column of $\mathscr{H}_L(w_\mathrm{d})$ is L-samples trajectory of $\mathscr{B}$

Follow-up questions for self-work

what is the meaning of image $\mathscr{H}_L(w_\mathrm{d})$?

how to use it in practice, *e.g.*, for simulation?

try it out on a numerical example?

Comparison of the heuristic methods for dealing with noise on simulated data

implement (SOL) and try it on an example

1. using exact data (random trajectory of a random system)
2. using noisy data $w_\text{d} = \overline{w} + \widetilde{w}$, where $\overline{w} \in \mathscr{B} \in \mathscr{L}^q$

modify the solution of (SOL)

1. using the pseudo-inverse
2. using SVD truncation, imposing rank $\mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B})$
3. ℓ_1-norm using regularization

comment on the results