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Hyperaccuracy for Geometric Fitting

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1 Geometric Fitting

Geometric fitting is to fit to noisy data a geometric model (a curve, a surface, or a relationship in high dimensions) expressed as an *implicit* equation that noise free data should satisfy [5, 11]. Its major differences from the traditional parameter estimation problem are:

- Unlike traditional statistical problems, there is no *explicit* model which explains observed data in terms of deterministic mechanisms and random errors. All descriptions are *implicit*.
- There are no inputs or outputs. No such concepts as causes and effects exist. No such distinctions as ordinates and abscissas exist.
- The underlying data space is homogeneous and isotropic with no inherent coordinate systems. Hence, the estimation process and the results should be *invariant* to changes of the coordinate system with respect to which the data are described.
- In many cases, the data are geometrically constrained. Typically, they are points on curves, surfaces, and hypersurfaces inherent in the data (e.g., unit vectors and matrices of determinant 0). Often, the parameters to be estimated are also similarly constrained. Hence, the *Gaussian distribution*, the most fundamental noise modeling, does not exist in its strict sense in such constrained spaces.

This type of problem plays a central role in computer vision applications. While in the traditional domain of statistics, the *total least-squares method* and the *errors-in-variable model* are rather abnormal concepts, to which attention is paid as a special research theme, the above mentioned properties are the *norm* in many computer vision problems. Hence, one needs an independent mathematical framework for it, rather than importing results from the literature of traditional statistics. Yet, many statisticians and computer vision researchers seem still not fully aware of this.

2 Performance Evaluation

Due to the above mentioned characteristics, performance evaluation of geometric fitting algorithms is very different from that in the traditional domain of statistics [5, 11].

Traditional estimation

It is customary to evaluate estimation methods by investigating the asymptotic performance as the number of data increases, and the *consistency* is one of the main concerns of statisticians. This is based on the tenet of statistics that random disturbances can be overcome by sampling many data, invoking the law of large numbers and the central limit theorem.

This is also reasonable in practice, since methods whose performance grows rapidly as the number of data increases are preferable, because such methods can reach admissible accuracy with a fewer number of data than other methods (Fig. 1(a)). This is very desirable, since data sampling often entails high cost, which is the major obstacle to doing statistical estimation in the real world.

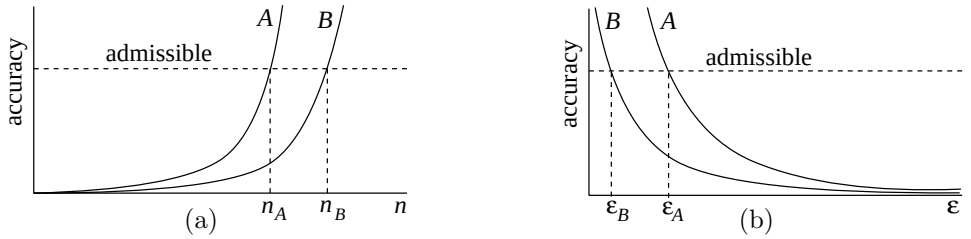

Figure 1: (a) For traditional statistical estimation, it is desired that the accuracy increases rapidly as $n \to \infty$ for the number n of sampling data, because admissible accuracy can be reached with a smaller number of sampling data. (b) For geometric fitting, it is desired that the accuracy increases rapidly as $\varepsilon \to 0$ for the noise level ε, because larger data uncertainty can be tolerated for admissible accuracy.

Geometric fitting

The data in computer vision applications are usually generated by a computer using image processing operations. They may contain pixel-level or subpixel-level errors, because image processing operations, often heuristically designed, are executed on digitized images and the results are not expected to be strictly correct.

In such small noise domains, it is reasonable to focus on the asymptotic performance as the noise level approaches 0 [5, 2, 11], since methods whose performance grows rapidly as the noise decreases can tolerate higher uncertainty in the data than others for admissible accuracy (Fig. 1(b)).

Duality

Thus, the mathematical framework for geometric fitting is in a sense *dual* to that of traditional statistic estimation. In this light, it can be shown that the *KCR* (*Kanatani-Cramer-Rao*) *lower bound* [6, 2, 10, 11] corresponds to the CR (Cramer-Rao) lower bound for traditional statistical estimation, and the *geometric AIC* [7] and the *geometric MDL* [9] correspond to the AIC (Akaike information criterion) [1] and the MDL (minimum description length) [14], respectively, for model selection in the traditional domain.

traditional	geometric
CR lower bound	KCR lower bound
AIC	geometric AIC
MDL	geometric MDL

3 Geometric ML

Since ML (maximum likelihood) estimation plays a central role for traditional parameter estimation, it is natural to do ML for geometric fitting, too. For this, however, some conceptual and technical difficulties exist. First of all, there is no explicit model that expresses the observed data in terms of parameters and noise terms, so the likelihood function cannot be explicitly defined. Also, the "Gaussian" distribution, the standard assumption for ML, cannot be defined over non-flat manifolds on which the data are constrained. Moreover, such elementary quantities as "means" and "covariances" cannot be defined (e.g., how can we define the mean and the covariance of points on a sphere?).

These difficulties can be resolved if we exploit the fact that noise is very small (pixel or subpixel level) in most computer vision applications. So, we can assume that noise is localized, and its distribution can be regarded as occurring in the "tangent space" to the manifold of the data domain at their true position. Then, we can indirectly define the likelihood function via Taylor expansion of the constraint given as an implicit equation. We call maximization of thus defined likelihood function *geometric ML*, or simply *ML* if there is no danger of confusion [5].

In many computer vision applications, the constraint can be written as a linear form in the parameters to be estimated by (nonlinearly) changing variables and embedding them in a high dimensional space. For such linearized constraints, we can easily compute the (geometric) ML estimator. The best known are:

- The *renormalization* method of Kanatani [4, 5, 8, 11].
- The *HEIV* (*heteroscedastic errors in variable*) method of Leedan and Meer [12].
- The *FNS* (*fundamental numerical scheme*) of Chojnacki et al. [3].

All of these are optimal in the sense that the solution satisfies KCR lower bound except for high order terms in the noise level [2, 11].

4 Hyperaccuracy

For simplicity of description, let us call those estimation methods who have accuracy equivalent to ML *high accuracy* methods and those whose accuracy are lower *low accuracy methods*, typical examples being least squares, the method of Taubin et al. [15, 16], and the method of Mühlich and Mester [13]. Finally, we call those methods who perform better than ML *hyperaccuracy methods*.

low accuracy	least squares, Taubin et al., Mühlich and Mester
high accuracy	renormalization, HEIV, FNS
hyperaccuracy	proposed

In this presentation, we demonstrate the existence of a hyperaccurate method. Since the ML estimator achieves the KCR lower bound except for high order noise terms, the difference of hyperaccurate solution from the ML estimator ought to be in higher order noise terms and hence is necessarily very small. Nevertheless, the underlying principle for obtainig such a method is theoretically very important, illuminating the relationship between geometric ML and the KCR lower bound.

The basic principle for obtaining hyperaccurate methods lies in the departure from "minimization". Statistical optimization has traditionally been associated with minimizing some cost function. Different cost functions have been proposed by different researchers, and the accuracy of their minimizing solutions have been meticulously analyzed and experimentally tested using synthetic data.

Actually, however, there is *no need* to consider cost functions at all. Since we have the KCR lower bound at hand, all we need to do is directly devise an estimator whose deviations from the true value satisfy the KCR lower bound on average. The renormalization of Kanatani was derived by this strategy, not minimizing anything. We show that by extending this principle to higher order terms, we can arrive at a "hyper accuracy" method.

We take ellipse fitting as an example and experimentally demonstrate how our hyperaccuracy method outperforms ML (Fig. 2). We discuss its problems and inherent limitations at the same time.

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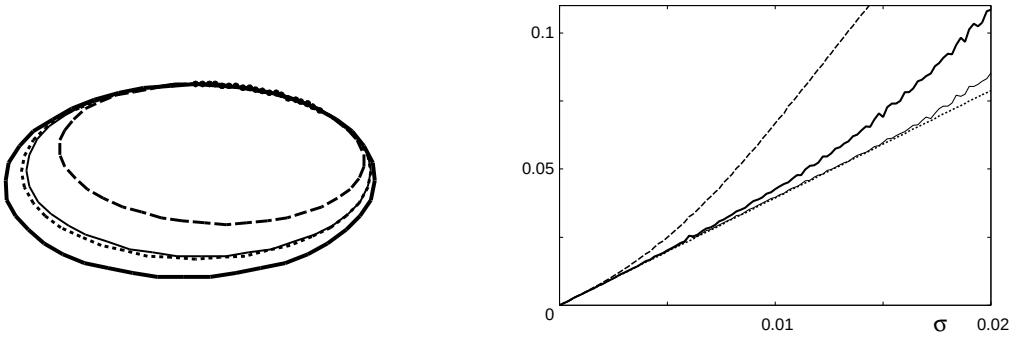

(a) (b)

Figure 2: (a) An instances of ellipse fitting: LS (broken line), ML (thick solid line), hyperaccuracy correction (thin solid line), true ellipse (dotted line). (b) Noise level vs. fitting error: LS (broken line), ML (thick solid line), hyperaccuracy correction (thin solid line), KCR lower bound (dotted line).

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