# On the Role of Constraints in System Identification

Arie Yeredor
School of Electrical Engineering
Dept. of Electrical Engineering - Systems
Tel-Aviv University, Israel
arie@eng.tau.ac.il

## I. The general framework

System identification is concerned with the estimation of parameters characterizing an unknown system. The estimation is usually based on observations of the system's (possibly noisy) input(s) and output(s). In the so-called "blind" system identification scenario, the estimation is based on the observed output(s) only, aided by some general knowledge about statistical properties of the input(s), rather than by actual observations thereof.

Quite commonly, the discussion is limited to discrete-time systems, assumed to be linear and time-invariant (LTI), stable and causal. As such, their input-output relation can always[1] be described as

$$y[t] = \sum_{\ell=0}^{\infty} h[\ell]u[t-\ell] \quad \forall t \in \mathbb{Z}, \tag{1}$$

where $u[t]$ is the input, $y[t]$ is the output and $h[\ell]$ is the system's impulse response. In the Multiple-Inputs, Multiple-Outputs (MIMO) case, the inputs and outputs may assume a vector form, but the basic form of the convolutive relation remains the same:

$$\boldsymbol{y}[t] = \sum_{\ell=0}^{\infty} \boldsymbol{H}[\ell]\boldsymbol{u}[t-\ell] \quad \forall t \in \mathbb{Z}, \tag{2}$$

where $\boldsymbol{u}[t]$ and $\boldsymbol{y}[t]$ denote (respectively) $M$-dimensional and $L$-dimensional input and output vectors, and $\boldsymbol{H}[\ell]$ denote the $L \times M$ impulse response matrices.

Although such systems are fully described by their (generally infinite) impulse response, some prior knowledge pertaining to their structure often allows to assume that they can also be described by a reduced (finite) set of parameters. For example, the Single-Input Single-Output (SISO) model (1) is often also modeled by a difference equation,

$$a_0 y[t] = -\sum_{k=1}^{N_p} a_k y[t-k] + \sum_{k=1}^{N_z} b_k u[t-k], \tag{3}$$

where $N_p$ and $N_z$ are (respectively) the number of poles and zeros in this model and $\boldsymbol{\theta} \triangleq [a_0\ a_1\ a_2\ \cdots\ a_{N_p}\ b_0\ b_1\ \cdots\ b_{N_z}]^T$ is the finite vector of unknown system's parameters.

Likewise, the MIMO model (2) is often described using a State-Space model,

$$\begin{aligned} \boldsymbol{x}[t+1] &= \boldsymbol{A}\boldsymbol{x}[t] + \boldsymbol{B}\boldsymbol{u}[t] \\ \boldsymbol{y}[t] &= \boldsymbol{C}\boldsymbol{x}[t] + \boldsymbol{D}\boldsymbol{u}[t], \end{aligned} \tag{4}$$

[1] barring the usually uninteresting possibility of an additive constant.

where $\boldsymbol{x}[t]$ is an "internal" (unobserved) $N$-dimensional state-vector, and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are matrices of the appropriate dimensions, which together comprise the finite set of unknown model parameters $\boldsymbol{\theta}$.

Similarly, description of the MIMO system with a (matrix) difference equation model, or of the SISO system with a state-space model are also possible. We shall regard the SISO case as a particular case of the MIMO case, except where the distinction is necessary.

## II. IDENTIFICATION VIA CONSTRAINED OPTIMIZATION

Assume that an observation interval of length $T$ is available. Typical system identification approaches seek to minimize (or to maximize) some criterion, which generally involves all of the available output/input observations $\boldsymbol{Y} \stackrel{\triangle}{=} [\boldsymbol{y}[1]\ \boldsymbol{y}[2]\ \cdots\ \boldsymbol{y}[T]]$ and $\boldsymbol{U} \stackrel{\triangle}{=} [\boldsymbol{u}[1]\ \boldsymbol{u}[2]\ \cdots\ \boldsymbol{u}[T]]$ (which is absent in the blind scenario), the unknown system parameters $\boldsymbol{\theta}$, and possibly some additional "nuisance parameters" $\boldsymbol{\phi}$, often representing some underlying, unobserved signals. Optimization of the criterion is sought with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$, yielding in turn the estimates of these parameters:

$$\min_{\boldsymbol{\theta},\boldsymbol{\phi}} C(\boldsymbol{Y},\boldsymbol{U};\boldsymbol{\theta},\boldsymbol{\phi}) \quad \Rightarrow \quad \hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}. \tag{5}$$

Often, however, some constraints on either $\boldsymbol{\theta}$, $\boldsymbol{\phi}$ or both are introduced into the optimization (5). The motivation for incorporating these constraints can come from a surprisingly large variety of perspectives on the problem. The main goal of this paper is to review the different approaches that lead to different types of constraints, each with the associated motivations, and to outline the resulting optimization and estimation approaches, providing some comparative study of the results.

Following are a few examples of useful constraints, with brief description of the motivation and frameworks by which they are applied.

- Constraints aimed at avoiding a trivial minimizer of the criterion. This is usually the basic motivation for adding constraints, where the associated optimization problem cannot yield a useful solution without excluding trivial solutions from the feasibility set.
- Constraints aimed at incorporating prior knowledge about the system, such as the locations of some of its poles or zeros (in a fashion similar to [4], [2]), so as to improve the resulting estimation accuracy by effectively reducing the number of degrees of freedom.
- Constraints aimed at imposing certain "natural" structures on some of the signals involves. This type of constraints usually involves the nuisance parameters $\boldsymbol{\phi}$, rather than the parameters of interest $\boldsymbol{\theta}$. For example, in the Structured Total Least Squares (STLS, e.g., [8]), $\boldsymbol{\phi}$ can consist of the estimated noiseless signal matrix, whereas the constraints confine the elements of that matrix to obey a certain structure (Hankel, Toeplitz, etc.).
- Constraints aimed at mitigating the bias induced by additive output noise (e.g., [9], [7], [16]) or by the use of an inconsistent criterion [18].
- Constraints aimed at guaranteeing the stability of the resulting estimated system [5], [9].

Most of these constraints would take the form of "equality constraints", namely $\boldsymbol{f}(\boldsymbol{\theta},\boldsymbol{\phi}) = \boldsymbol{0}$ (where $\boldsymbol{f}(\cdot)$ is an associated vector function), but some may also involve forms of inequalities. The basic approaches for optimizing the criteria under the associated types of constraints (Lagrange multipliers, Method of multipliers, Successive projections, Linear programming) will also be reviewed and graphically illustrated, where applicable.

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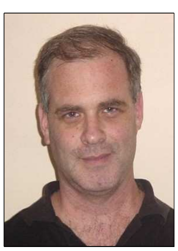


**Arie Yeredor** received the B.Sc. *(summa cum laude)* and Ph.D. degrees in electrical engineering from Tel-Aviv University (TAU), Tel Aviv, Israel, in 1984 and 1997, respectively. He is currently a Senior Lecturer with the Department of Electrical Engineering - Systems, at TAU's School of Electrical Engineering, where he teaches courses in statistical and digital signal processing, and has been awarded the yearly Best Lecturer of the Faculty of Engineering Award five times. He also holds a consulting position with NICE Systems Inc., Raanana, Israel, in the fields of speech and audio processing, video processing, and emitter location algorithms. Dr. Yeredor is an Associate Editor for IEEE SIGNAL PROCESSING LETTERS and for IEEE TRANSACTIONS ON CIRCUITS AND SYSTEM II, and is a member of the IEEE Signal Processing Society's Signal Processing Theory and Methods (SPTM) Technical Committee.