Consistent estimation of regression coefficients in measurement error model under exact linear restrictions

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- **To analyze the effects of non-normality of measurement errors on the estimators.**

- **To obtain the dominance conditions on efficiency properties of the estimators.**

Outline of the Talk

The Model

Consider the multivariate linear ultrastructural model

$$\eta = \boldsymbol{\xi}'\boldsymbol{\beta},$$

where

η : true study variable

$\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots \xi_p)'$, true explanatory vector

$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)'$, vector of regression coefficients

Also,

$$y = \eta + \epsilon \text{ and } \mathbf{x} = \boldsymbol{\xi} + \boldsymbol{\delta},$$

where

y : observed study variable

ϵ : measurement error associated with study variable

$\mathbf{x}$: observed explanatory vector

$\boldsymbol{\delta}$: measurement error vector associated with explanatory vector

For a sample of size n, the model is formulated as

$$\begin{aligned} \boldsymbol{\eta} &= T\boldsymbol{\beta}, \\ \mathbf{y} &= \boldsymbol{\eta} + \boldsymbol{\epsilon}, \\ X &= T + \Delta, \end{aligned}$$

where

$\boldsymbol{\eta} = (\eta_i)_{n\times 1}$, vector of true study variables,
$\mathbf{y} = (y_i)_{n\times 1}$, vector of observed study variables,
$T = (\xi_{ij})_{n\times p}$, matrix of true explanatory variables,
$X = (x_{ij})_{n\times p}$, matrix of observed explanatory variables,
$\Delta = (\delta_{ij})_{n\times p}$, matrix of measurement errors associated with ξ_{ij} and
$\boldsymbol{\epsilon} = (\epsilon_i)_{n\times 1}$, vector of measurement errors associated with y_i.

Assume that ξ_{ij}'s are independently distributed with

$$E(\xi_{ij}) = \mu_{ij}.$$

We can write

$$\xi_{ij} = \mu_{ij} + \phi_{ij},$$

$$(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$$

where ϕ_{ij} is the random disturbance term with $E(\phi_{ij}) = 0$.

Define,
$M := (\mu_{ij})$ and $\Phi := (\phi_{ij})$ are $n \times p$ matrices.

So we can write,

$$T = M + \Phi.$$

Exact Linear Restrictions

Suppose, there are $J(< p)$ exact linear restrictions on the regression coefficients $(\beta_1, \beta_2, \ldots \beta_p)$ given as

$$r = R\boldsymbol{\beta},$$

where
r is a $J \times 1$ known vector and
R is a $J \times p$ known matrix of full row rank.

Assumptions

- $\delta_{ij} \stackrel{iid}{\sim} \left(0,\, \sigma_\delta^2,\, \gamma_{1\delta}\sigma_\delta^3,\, (\gamma_{2\delta}+3)\sigma_\delta^4\right)$,
- $\phi_{ij} \stackrel{iid}{\sim} \left(0,\, \sigma_\phi^2,\, \gamma_{1\phi}\sigma_\phi^3,\, (\gamma_{2\phi}+3)\sigma_\phi^4\right)$,
- $\epsilon_i \stackrel{iid}{\sim} \left(0,\, \sigma_\epsilon^2,\, \gamma_{1\epsilon}\sigma_\epsilon^3,\, (\gamma_{2\epsilon}+3)\sigma_\epsilon^4\right)$,
  (where $\gamma_{1\cdot}$: coefficients of skewness and $\gamma_{2\cdot}$: coefficients of kurtosis.)
- $\boldsymbol{\epsilon}$, Δ and Φ are statistically independent,
- $\lim_{n\to\infty} \frac{1}{n} M'M =: \Sigma_\mu$ exists and is nonsingular,
- $\lim_{n\to\infty} \frac{1}{n} M'e_n =: \boldsymbol{\sigma}_\mu$ exists and is finite,
  (where $e_n = (1, 1, \ldots, 1)'_{n\times 1}$).

{No specific distributional form of any random quantity is assumed.}

Usual Estimators

Define, $S := X'X$ and $\Sigma := \Sigma_\mu + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p$.

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Define, $S := X'X$ and $\Sigma := \Sigma_\mu + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p$.

(1) Ordinary least squares estimator (OLSE)

$$b = S^{-1}X'y,$$

we have, $\text{plim}\ (b - \beta) = -\sigma_\delta^2 \Sigma^{-1} \beta \neq 0$

$$(= 0, \text{ when } \sigma_\delta^2 = 0)$$

inconsistent

and $Rb \neq r$.

doesn't satisfy the restrictions

(2) Restricted least squares estimator (RLSE)

$$b_R = b + S^{-1}R'(RS^{-1}R')^{-1}(r - R\beta),$$

we have, $\text{plim}\ (b_R - \beta) = -\sigma_\delta^2 \Sigma^{-1}(I_p - R'(R\Sigma^{-1}R')^{-1}R\Sigma^{-1})\beta \neq 0$

$(= 0,\ \text{when}\ \sigma_\delta^2 = 0\ \text{or}\ J = p)$, inconsistent

however, $Rb_R = r$. satisfies the restrictions

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$(= 0,$ when $\sigma_\delta^2 = 0$ or $J = p)$, inconsistent

however, $Rb_R = r$. satisfies the restrictions

(3) Adjusted least squares estimator (when $\text{cov}(\boldsymbol{\delta}) = \sigma_\delta^2 I_p$ is known)

$$b_\delta^{(1)} = (S - n\sigma_\delta^2 I_p)^{-1}X'y,$$

we have, plim $b_\delta^{(1)} - \boldsymbol{\beta} = 0$, consistent

however, $Rb_\delta^{(1)} \neq r$. doesn't satisfy the restrictions

Problem:
To find such an estimator of β, which is **consistent** as well as **satisfies the restrictions** $R\beta = r$.

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Known Result:
Some information about the unknown parameters is needed for consistent estimation in measurement errors models.

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Known Result:
Some information about the unknown parameters is needed for consistent estimation in measurement errors models.

Additional Information:
We use the knowledge of $\text{cov}(\boldsymbol{\delta}) = \sigma_\delta^2 I_p$ (or equivalently σ_δ^2).

Consistent estimators satisfying the restrictions

(1) First estimator is obtained

- by replacing b in b_R by $b_\delta^{(1)}$ or equivalently
- by minimizing $(b_\delta^{(1)} - \boldsymbol{\beta})' S (b_\delta^{(1)} - \boldsymbol{\beta})$ subject to $R\beta = r$.

This estimator is given by

$$b_\delta^{(2)} = b_\delta^{(1)} + S^{-1} R' R_S^{-1} (r - R b_\delta^{(1)}),$$

where $R_S = RS^{-1}R'$.

We see,

$$\begin{aligned} \text{plim } b_\delta^{(2)} &= \boldsymbol{\beta}, \\ R b_\delta^{(2)} &= r. \end{aligned}$$

consistent and satisfies the restrictions

(2) Since $\text{plim}(\frac{1}{n}S) = \Sigma$,

another consistent estimator of $\boldsymbol{\beta}$ is obtained by adjusting the inconsistency in b_R.

This estimator is

$$b_\delta^{(3)} = [I_p - n\sigma_\delta^2(I_p - S^{-1}R'R_S^{-1}R)S^{-1}]^{-1}b_R.$$

We see,

$$\begin{aligned} \text{plim } b_\delta^{(3)} &= \boldsymbol{\beta}, \\ Rb_\delta^{(3)} &= r. \end{aligned}$$

consistent and satisfies the restrictions

(3) Another estimator is obtained by minimizing

$$(b_\delta^{(1)} - \boldsymbol{\beta})'(b_\delta^{(1)} - \boldsymbol{\beta})$$

with respect to $\boldsymbol{\beta}$ subject $R\boldsymbol{\beta} = r$.

This estimator is given by

$$b_\delta^{(4)} = b_\delta^{(1)} + R'(RR')^{-1}(r - Rb_\delta^{(1)}).$$

We see ,

$$\begin{aligned} \text{plim } b_\delta^{(4)} &= \boldsymbol{\beta}, \\ Rb_\delta^{(4)} &= r. \end{aligned}$$

consistent and satisfies the restrictions

Asymptotic Properties

Define $D := \sigma_\phi^2 \Sigma_T^{-1}$,
where $\Sigma_T = \frac{1}{n} M'M + \sigma_\phi^2 Ip$.

In functional form, $\sigma_\phi^2 = 0 \Rightarrow D = 0$.

In the structural form, $M = 0 \Rightarrow D = I_p$.

D : Measure of departure of the ultrastructural form from functional and structural forms.

Asymptotic Properties

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D : Measure of departure of the ultrastructural form from functional and structural forms.

Define $\Theta := \sigma_\delta^2 \Sigma_X^{-1}$,
where $\Sigma_X = \frac{1}{n} M'M + \sigma_\phi^2 Ip + \sigma_\delta^2 Ip$.

In case of classical regression, when measurement errors are absent, $\sigma_\delta^2 = 0 \Rightarrow \Theta = 0$.

Θ : Measure of departure of the classical regression model from the measurement error model.

Theorem

The large sample asymptotic bias of $b_\delta^{(2)}$, $b_\delta^{(3)}$ *and* $b_\delta^{(4)}$ *are*

$$
\begin{aligned}
Bias(b_\delta^{(2)}) &= \frac{1}{n\sigma_\phi^2}(I_p - B\Theta^{-1})[\sigma_\delta^2 D\Theta^{-1}(\sigma_\delta^2 D - \sigma_\phi^2 B) \\
&\quad + \sigma_\delta^2(tr\ \frac{1}{n}MDM')D - \sigma_\phi^2(tr\ \frac{1}{n}MB\Theta^{-1}DM')\Theta \\
&\quad + (\sigma_\delta^2 + \sigma_\phi^2)\{\sigma_\delta^2(tr\ D)D - \sigma_\phi^2(tr\ B\Theta^{-1}D)\Theta\} \\
&\quad + N_{1\delta}]\beta \quad upto\ O(n^{-1}), \\
Bias(b_\delta^{(3)}) &= 0 \quad upto\ O(n^{-\frac{1}{2}}), \\
Bias(b_\delta^{(4)}) &= \frac{\sigma_\delta^2}{n\sigma_\phi^4}[I_p - R'(RR')^{-1}R]D[\sigma_\delta^2\Theta^{-1}D + (\sigma_\delta^2 + \sigma_\phi^2)(tr\ D)I_p \\
&\quad + (tr\ \frac{1}{n}MDM')I_p + N_{2\delta}]\beta \quad upto\ O(n^{-1}).
\end{aligned}
$$

where

$$
\begin{aligned}
B &= \Theta R'(R\Theta R')^{-1}R\Theta, \\
N_{1\delta} &= \gamma_{1\delta}\sigma_\delta\big[\sigma_\delta^2\big\{f(Ds_\mu e_p', D) + 2f(D, s_\mu e_p' D)\big\} \\
&\quad -\sigma_\phi^2\big\{f(\Theta s_\mu e_p', B\Theta^{-1}D) + 2f(\Theta, s_\mu e_p' B\Theta^{-1}D)\big\}\big] \\
&\quad +\gamma_{2\delta}\sigma_\delta^2\big[\sigma_\delta^2 f(D, D) - \sigma_\phi^2 f(\Theta, B\Theta^{-1}D)\big], \\
N_{2\delta} &= \gamma_{1\delta}\sigma_\delta\big\{f(s_\mu e_p', D) + 2f(I_p, e_p s_\mu' D)\big\} + \gamma_{2\delta}\sigma_\delta^2 f(I_p, D), \\
s_\mu &= \frac{1}{n}M'e_n,
\end{aligned}
$$

$f(Z_1, Z_2) = Z_1(Z_2 * I_p),\ Z_1, Z_2 \in \mathbb{R}^{p\times p}$

* : Hadamard product (elementwise product) operator of matrices

$N_{1\delta}$ and $N_{2\delta}$: Contribution of non-normality of error distributions in the bias of $b_\delta^{(l)}$, $(l = 2, 3, 4)$

Bias of $b_\delta^{(3)}$ upto order $O(n^{-1})$ is quite complicated.

All the estimators $b_\delta^{(j)}$ $(j = 2, 3, 4)$ are unbiased upto $O(n^{-1/2})$.

The bias of $b_\delta^{(2)}$ and $b_\delta^{(4)}$:

- affected only by the skewness and kurtosis of the distributions of δ_{ij}'s
- non-normality of the distributions of ϵ_i's and ϕ_{ij}'s has no role to play at least upto $O(n^{-1})$

Theorem

The large sample approximations of mean squared error matrices (MSEM) of $b_\delta^{(2)}$, $b_\delta^{(3)}$ and $b_\delta^{(4)}$ upto $O(n^{-1})$ are

$$
\begin{aligned}
MSEM(b_\delta^{(2)}) &= \frac{\sigma_\delta^2}{n\sigma_\phi^4}(I_p - B\Theta^{-1})D\big[\big(\sigma_\epsilon^2 + \sigma_\delta^2(tr\,\boldsymbol{\beta\beta}')\big)\Theta^{-1} \\
&\quad + \sigma_\delta^2\boldsymbol{\beta\beta}' + N_{3\delta}\big]D(I_p - \Theta^{-1}B), \\
MSEM(b_\delta^{(3)}) &= \frac{1}{n\sigma_\delta^2}[I_p - (\Theta - B)]^{-1}(\Theta - B)\big[\big(\sigma_\epsilon^2 + \sigma_\delta^2(tr\,\boldsymbol{\beta\beta}')\big)\Theta^{-1} \\
&\quad + \sigma_\delta^2\boldsymbol{\beta\beta}' + N_{3\delta}\big](\Theta - B)[I_p - (\Theta - B)]^{-1}, \\
MSEM(b_\delta^{(4)}) &= \frac{\sigma_\delta^2}{n\sigma_\phi^4}[I_p - R'(RR')^{-1}R]D\big[\big(\sigma_\epsilon^2 + \sigma_\delta^2(tr\,\boldsymbol{\beta\beta}')\big)\Theta^{-1} \\
&\quad + \sigma_\delta^2\boldsymbol{\beta\beta}' + N_{3\delta}\big]D[I_p - R'(RR')^{-1}R],
\end{aligned}
$$

where $N_{3\delta} = \gamma_{1\delta}\sigma_\delta[f(s_\mu e_p', \boldsymbol{\beta\beta}') + \{f(s_\mu e_p', \boldsymbol{\beta\beta}')\}'] + \gamma_{2\delta}\sigma_\delta^2 f(I_p, \boldsymbol{\beta\beta}')$.

$N_{3\delta}$: Contribution of departure from normality on the MSE matrices of these estimators

MSE matrices of the estimators $b_\delta^{(j)}$ $(j = 2, 3, 4)$:

- affected by the skewness and kurtosis of the distribution of δ_{ij}'s
- no non-normality effect of ϵ_i's and ϕ_{ij}'s

Asymptotic distributions of the estimators

Theorem

As $n \to \infty$,

$$\sqrt{n}\left(b_\delta^{(j)} - \boldsymbol{\beta}\right) \sim N_p(0,\ A_j W A_j'),$$

$(j = 2, 3, 4)$, *where,*

$$\begin{aligned}
W &= \sigma_\epsilon^2 \Sigma + \sigma_\delta^2 (tr\ \boldsymbol{\beta}\boldsymbol{\beta}')\Sigma + \sigma_\delta^4 \boldsymbol{\beta}\boldsymbol{\beta}' \\
&\quad + \gamma_{1\delta}\sigma_\delta^3 \left\{ f(\boldsymbol{\sigma}_\mu e_p', \boldsymbol{\beta}\boldsymbol{\beta}') + \left(f(\boldsymbol{\sigma}_\mu e_p', \boldsymbol{\beta}\boldsymbol{\beta}') \right)' \right\} + \gamma_{2\delta}\sigma_\delta^4 f(I_p, \boldsymbol{\beta}\boldsymbol{\beta}), \\
A_2 &= \left(I_p - Q\Sigma \right)(\Sigma - \sigma_\delta^2 I_p)^{-1}, \\
A_3 &= \left\{ I_p - \sigma_\delta^2 (\Sigma^{-1} - Q) \right\}^{-1} (\Sigma^{-1} - Q), \\
A_4 &= [I_p - R'(RR')^{-1}R](\Sigma - \sigma_\delta^2 I_p)^{-1}, \\
Q &= \Sigma^{-1} R' (R\Sigma^{-1}R')^{-1} R\Sigma^{-1}.
\end{aligned}$$

Remarks

Covariance matrix of asymptotic distribution of $\sqrt{n}(b_\delta^{(j)} - \beta)$,

- $A_j W A_j' = \text{function}(\gamma_{1\delta},\ \gamma_{2\delta})$, while
  $A_j W A_j' \neq \text{function}(\gamma_{1\phi},\ \gamma_{2\phi},\ \gamma_{1\epsilon},\ \gamma_{2\epsilon})$.
- $A_j W A_j' = \lim_{n\to\infty} n.\text{MSEM}\ (b_\delta^{(j)})$,

$(j = 2, 3, 4.)$

Dominance conditions of the estimators

Theorem

If $(A_j{}'A_j - A_k{}'A_k)$ is a non-negative definite matrix then for every $\alpha > 0$,

$$\lim_{n\to\infty} P\{\|\sqrt{n}(b_\delta^{(j)} - \beta)\| \leq \alpha\} \leq \lim_{n\to\infty} P\{\|\sqrt{n}(b_\delta^{(k)} - \beta)\| \leq \alpha\}, \qquad (1)$$

for $j, k = 2, 3, 4;\ j \neq k$ and

A_2, A_3 and A_4 are defined in previous theorem.

Remarks

- $\lim_{n\to\infty} P\{\|\sqrt{n}(b_\delta^{(j)} - \beta)\| \le \alpha\} \le \lim_{n\to\infty} P\{\|\sqrt{n}(b_\delta^{(k)} - \beta)\| \le \alpha\}$
  - means that $b_\delta^{(k)}$ is closer to β than $b_\delta^{(j)}$ in stochastic order.
  - implies that for any increasing function $h(\cdot)$,
    $E[h(b_\delta^{(j)} - \beta)] \le E[h(b_\delta^{(k)} - \beta)]$, as $n \to \infty$

$j, k = 2, 3, 4;\ j \ne k.$

Remarks

- $\lim_{n\to\infty} P\{\|\sqrt{n}(b_\delta^{(j)} - \beta)\| \le \alpha\} \le \lim_{n\to\infty} P\{\|\sqrt{n}(b_\delta^{(k)} - \beta)\| \le \alpha\}$
  - means that $b_\delta^{(k)}$ is closer to β than $b_\delta^{(j)}$ in stochastic order.
  - implies that for any increasing function $h(\cdot)$,
    $E[h(b_\delta^{(j)} - \beta)] \le E[h(b_\delta^{(k)} - \beta)],$ as $n \to \infty$

  $j, k = 2, 3, 4;\ j \ne k.$

- Since plim $\frac{1}{n}X'X = \Sigma$, A_2, A_3 and A_4 can be estimated consistently. This follows that the condition

  "$(A_j{}'A_j - A_k{}'A_k)$ is a non-negative definite matrix"

  can be checked for the sample.
  (true for sample $\Longrightarrow$ true for population for sufficiently large n.)

Monte-Carlo Simulation

The following values are used in the simulation-

$n = 25,\ 45;$

$$\beta = \begin{pmatrix} 2.2 \\ 1.1 \\ 3.0 \\ 4.2 \\ 2.5 \end{pmatrix}, \begin{pmatrix} -2.2 \\ -1.1 \\ -3.0 \\ -4.2 \\ -2.5 \end{pmatrix}, \begin{pmatrix} 2.2 \\ -1.1 \\ 3.0 \\ -4.2 \\ 2.5 \end{pmatrix};$$

$$\text{and } R = \begin{pmatrix} 0.8 & 0.6 & 0.7 & 0.9 & 0.8 \\ 0.2 & 0.7 & 0.4 & 0.7 & 0.8 \\ 0.6 & 0.4 & 0.6 & 0.1 & 0.4 \\ 0.5 & 0 & 0.8 & 0.9 & 0.4 \end{pmatrix}.$$

We choose a matrix M of mean values. For our chosen M,

$$\frac{1}{n}M'M = \begin{pmatrix} 10.1044 & 6.5674 & 7.3198 & 5.2212 & 8.5015 \\ 6.5674 & 8.9555 & 6.1732 & 5.4899 & 7.8445 \\ 7.3198 & 6.1732 & 8.1935 & 4.4971 & 7.2151 \\ 5.2212 & 5.4899 & 4.4971 & 6.2650 & 5.8162 \\ 8.5015 & 7.8445 & 7.2151 & 5.8162 & 10.9155 \end{pmatrix}, \text{ when } n = 25$$

and

$$\frac{1}{n}M'M = \begin{pmatrix} 8.9820 & 5.6047 & 6.5244 & 5.7537 & 7.2039 \\ 5.6047 & 7.5345 & 5.3032 & 5.7951 & 6.4435 \\ 6.5244 & 5.3032 & 7.7700 & 5.6016 & 6.3553 \\ 5.7537 & 5.7951 & 5.6016 & 7.8737 & 6.0282 \\ 7.2039 & 6.4435 & 6.3553 & 6.0282 & 9.4280 \end{pmatrix}, \text{ when } n = 45,$$

Replications = 5000.

To study the effect of skewness and kurtosis of the distributions of the random quantities in the model, we adopt the following distributions

(i) Normal distribution (having no skewness and no kurtosis),

(ii) Student's t distribution (having kurtosis only) and

(iii) Gamma Distribution (having both skewness and kurtosis).

Empirical bias of the estimators

Table I

$$\left(\epsilon_i \sim N(0,\, \sigma_\epsilon^2),\ \phi_{ij} \sim N(0,\, \sigma_\phi^2),\ \delta_{ij} \sim N(0,\, \sigma_\delta^2)\right)$$

$\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$					
$n = 25$			$n = 45$		
EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)	EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)
0.0008	0.0006	0.0015	-0.0001	-0.0002	0.0005
-0.0071	-0.0059	-0.0134	0.0011	0.0019	-0.0042
-0.0019	-0.0016	-0.0037	0.0003	0.0005	-0.0011
-0.0030	-0.0025	-0.0056	0.0005	0.0008	-0.0017
0.0096	0.0079	0.0182	-0.0015	-0.0026	0.0056
$\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 1.0$					
$n = 25$			$n = 45$		
EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)	EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)
0.0014	0.0002	0.0026	0.0011	0.0002	0.0023
-0.0081	-0.0017	-0.0191	-0.0077	-0.0015	-0.0179
-0.0021	-0.0005	-0.0071	-0.0019	-0.0004	-0.0063
-0.0027	-0.0007	-0.0093	-0.0021	-0.0007	-0.0085
0.0096	0.0023	0.0304	0.0082	0.0020	0.0290

(EB: Emperical Bias)

Table I (Cont..)

$\left(\epsilon_i \sim N(0,\, \sigma_\epsilon^2),\ \phi_{ij} \sim N(0,\, \sigma_\phi^2),\ \delta_{ij} \sim N(0,\, \sigma_\delta^2)\right)$

$\sigma_\epsilon^2 = 1.0476,\ \sigma_\phi^2 = 1.0476,\ \sigma_\delta^2 = 1.0476$					
$n = 25$			$n = 45$		
EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)	EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)
-0.0010	-0.0003	-0.0040	0.0003	-0.0003	0.0015
0.0095	0.0026	0.0365	-0.0032	0.0013	-0.0142
0.0026	0.0007	0.0099	-0.0009	0.0005	-0.0039
0.0040	0.0011	0.0153	-0.0013	0.0007	-0.0059
-0.0129	-0.0035	-0.0492	0.0043	-0.0018	0.0191

Table II

$\left(\epsilon_i \sim t_{(n-1)},\ \phi_{ij} \sim t_{(n-1)} \text{ and } \delta_{ij} \sim t_{(n-1)}\right)$

$\sigma_\epsilon^2 = 1.0476,\ \sigma_\phi^2 = 1.0476,\ \sigma_\delta^2 = 1.0476$					
$n = 25$			$n = 45$		
EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)	EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)
0.0013	-0.0005	0.0040	0.0006	-0.0003	0.0017
-0.0117	0.0035	-0.0369	-0.0043	0.0031	-0.0159
-0.0032	0.0010	-0.0101	-0.0011	0.0009	-0.0043
-0.0049	0.0014	-0.0155	-0.0019	0.0011	-0.0066
0.0158	-0.0047	0.0499	0.0057	-0.0035	0.0214

Table III

$$\left(\epsilon_i \sim \text{Gamma}(5, \sqrt{\sigma_\epsilon^2/5}),\ \phi_{ij} \sim \text{Gamma}(5, \sqrt{\sigma_\phi^2/5}) \text{ and } \delta_{ij} \sim \text{Gamma}(5, \sqrt{\sigma_\delta^2/5})\right)$$

$\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$					
$n = 25$			$n = 45$		
EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)	EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)
0.0010	0.0008	0.0016	-0.0007	-0.0006	-0.0009
-0.0088	-0.0077	-0.0147	0.0068	0.0070	0.0045
-0.0024	-0.0021	-0.0040	0.0018	0.0020	0.0011
-0.0037	-0.0032	-0.0062	0.0028	0.0029	0.0022
0.0118	0.0104	0.0199	-0.0091	-0.0099	-0.0057
$\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 1.0$					
$n = 25$			$n = 45$		
EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)	EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)
-0.0018	-0.0008	-0.0041	0.0008	0.0001	0.0004
0.0161	0.0046	0.0211	-0.0064	-0.0004	-0.0181
0.0047	0.0020	0.0073	-0.0025	-0.0008	-0.0054
0.0065	0.0015	0.0095	-0.0029	-0.0009	-0.0059
-0.0182	-0.0088	-0.0215	0.0152	0.0020	0.0118

Table III (Cont..)

$\left(\epsilon_i \sim \text{Gamma}(5,\ \sqrt{\sigma_\epsilon^2/5}),\ \phi_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\phi^2/5})\right.$ and

$\left.\delta_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\delta^2/5})\right)$

$\sigma_\epsilon^2 = 1.0476,\ \sigma_\phi^2 = 1.0476,\ \sigma_\delta^2 = 1.0476$					
$n = 25$			$n = 45$		
EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)	EB($b_\delta^{(2)}$)	EB($b_\delta^{(3)}$)	EB($b_\delta^{(4)}$)
0.0032	0.0007	0.0061	0.0007	0.0004	0.0017
-0.0289	-0.0068	-0.0556	-0.0067	-0.0032	-0.0157
-0.0079	-0.0018	-0.0152	-0.0018	-0.0009	-0.0043
-0.0121	-0.0028	-0.0233	-0.0028	-0.0013	-0.0066
0.0390	0.0091	0.0751	0.0090	0.0044	0.0212

It is observed from Tables I - III that

- all the bias are very small even for $n = 25$.
- bias magnitudes of $b_\delta^{(2)}$, $b_\delta^{(3)}$ and $b_\delta^{(4)}$ decrease as sample size increases.
- $b_\delta^{(3)}$ has the smallest absolute bias.
- $b_\delta^{(4)}$ has the largest absolute bias.
- due to change in sample size and values of various variances, the fluctuation in bias of $b_\delta^{(3)}$ is minimum.
- For Normal distribution and t distributions, the bias of all three estimators are almost same.
- For Gamma distribution, bias are large.

Empirical mean squared error matrices of the estimators

Table IV

$$\left(\epsilon_i \sim N(0,\, \sigma_\epsilon^2),\ \phi_{ij} \sim N(0,\, \sigma_\phi^2),\ \delta_{ij} \sim N(0,\, \sigma_\delta^2)\right)$$

$\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$										
	$n = 25$					$n = 45$				
$b_\delta^{(2)}$	0.0012	-0.0111	-0.0030	-0.0046	0.0150	0.0007	-0.0060	-0.0016	-0.0025	0.0081
	-0.0111	0.1019	0.0278	0.0426	-0.1375	-0.0060	0.0550	0.0150	0.0230	-0.0742
	-0.0030	0.0278	0.0076	0.0116	-0.0375	-0.0016	0.0150	0.0041	0.0063	-0.0202
	-0.0046	0.0426	0.0116	0.0178	-0.0575	-0.0025	0.0230	0.0063	0.0096	-0.0310
	0.0150	-0.1375	-0.0375	-0.0575	0.1857	0.0081	-0.0742	-0.0202	-0.0310	0.1002
$b_\delta^{(3)}$	0.0012	-0.0109	-0.0030	-0.0046	0.0147	0.0006	-0.0059	-0.0016	-0.0025	0.0080
	-0.0109	0.1000	0.0273	0.0418	-0.1349	-0.0059	0.0544	0.0148	0.0227	-0.0734
	-0.0030	0.0273	0.0074	0.0114	-0.0368	-0.0016	0.0148	0.0040	0.0062	-0.0200
	-0.0046	0.0418	0.0114	0.0175	-0.0564	-0.0025	0.0227	0.0062	0.0095	-0.0307
	0.0147	-0.1349	-0.0368	-0.0564	0.1822	0.0080	-0.0734	-0.0200	-0.0307	0.0991
$b_\delta^{(4)}$	0.0016	-0.0143	-0.0039	-0.0060	0.0193	0.0008	-0.0071	-0.0019	-0.0030	0.0095
	-0.0143	0.1308	0.0357	0.0547	-0.1766	-0.0071	0.0647	0.0176	0.0271	-0.0874
	-0.0039	0.0357	0.0097	0.0149	-0.0482	-0.0019	0.0176	0.0048	0.0074	-0.0238
	-0.0060	0.0547	0.0149	0.0229	-0.0739	-0.0030	0.0271	0.0074	0.0113	-0.0365
	0.0193	-0.1766	-0.0482	-0.0739	0.2384	0.0095	-0.0874	-0.0238	-0.0365	0.1179

Table V

$$\left(\epsilon_i \sim N(0,\, \sigma^2_\epsilon),\ \phi_{ij} \sim N(0,\, \sigma^2_\phi),\ \delta_{ij} \sim N(0,\, \sigma^2_\delta)\right)$$

$\sigma^2_\epsilon = 0.4,\ \sigma^2_\phi = 0.4,\ \sigma^2_\delta = 1.0$										
	$n = 25$					$n = 45$				
$b^{(2)}_\delta$	0.0111	-0.1020	-0.0278	-0.0426	0.1376	0.0021	-0.0196	-0.0054	-0.0082	0.0265
	-0.1020	0.9347	0.2549	0.3909	-1.2618	-0.0196	0.1798	0.0490	0.0752	-0.2428
	-0.0278	0.2549	0.0695	0.1066	-0.3441	-0.0054	0.0490	0.0134	0.0205	-0.0662
	-0.0426	0.3909	0.1066	0.1634	-0.5277	-0.0082	0.0752	0.0205	0.0314	-0.1015
	0.1376	-1.2618	-0.3441	-0.5277	1.7034	0.0265	-0.2428	-0.0662	-0.1015	0.3278
$b^{(3)}_\delta$	0.0040	-0.0370	-0.0101	-0.0155	0.0499	0.0020	-0.0180	-0.0049	-0.0075	0.0243
	-0.0370	0.3390	0.0925	0.1418	-0.4577	-0.0180	0.1653	0.0451	0.0691	-0.2231
	-0.0101	0.0925	0.0252	0.0387	-0.1248	-0.0049	0.0451	0.0123	0.0189	-0.0609
	-0.0155	0.1418	0.0387	0.0593	-0.1914	-0.0075	0.0691	0.0189	0.0289	-0.0933
	0.0499	-0.4577	-0.1248	-0.1914	0.6179	0.0243	-0.2231	-0.0609	-0.0933	0.3012
$b^{(4)}_\delta$	0.0588	-0.5390	-0.1470	-0.2254	0.7276	0.0030	-0.0273	-0.0074	-0.0114	0.0368
	-0.5390	4.9405	1.3474	2.0660	-6.6696	-0.0273	0.2499	0.0682	0.1045	-0.3374
	-0.1470	1.3474	0.3675	0.5635	-1.8190	-0.0074	0.0682	0.0186	0.0285	-0.0920
	-0.2254	2.0660	0.5635	0.8640	-2.7891	-0.0114	0.1045	0.0285	0.0437	-0.1411
	0.7276	-6.6696	-1.8190	-2.7891	9.0040	0.0368	-0.3374	-0.0920	-0.1411	0.4555

Table VI

$$\left(\epsilon_i \sim N(0,\, \sigma^2_\epsilon),\ \phi_{ij} \sim N(0,\, \sigma^2_\phi),\ \delta_{ij} \sim N(0,\, \sigma^2_\delta)\right)$$

$\sigma^2_\epsilon = 1.4076,\ \sigma^2_\phi = 1.4076,\ \sigma^2_\delta = 1.4076$										
	$n = 25$					$n = 45$				
$b^{(2)}_\delta$	0.0069	-0.0635	-0.0173	-0.0266	0.0858	0.0018	-0.0161	-0.0044	-0.0067	0.0217
	-0.0635	0.5823	0.1588	0.2435	-0.7861	-0.0161	0.1472	0.0401	0.0615	-0.1987
	-0.0173	0.1588	0.0433	0.0664	-0.2144	-0.0044	0.0401	0.0109	0.0168	-0.0542
	-0.0266	0.2435	0.0664	0.1018	-0.3287	-0.0067	0.0615	0.0168	0.0257	-0.0831
	0.0858	-0.7861	-0.2144	-0.3287	1.0612	0.0217	-0.1987	-0.0542	-0.0831	0.2682
$b^{(3)}_\delta$	0.0033	-0.0305	-0.0083	-0.0127	0.0412	0.0016	-0.0151	-0.0041	-0.0063	0.0204
	-0.0305	0.2795	0.0762	0.1169	-0.3773	-0.0151	0.1383	0.0377	0.0578	-0.1867
	-0.0083	0.0762	0.0208	0.0319	-0.1029	-0.0041	0.0377	0.0103	0.0158	-0.0509
	-0.0127	0.1169	0.0319	0.0489	-0.1578	-0.0063	0.0578	0.0158	0.0242	-0.0781
	0.0412	-0.3773	-0.1029	-0.1578	0.5093	0.0204	-0.1867	-0.0509	-0.0781	0.2521
$b^{(4)}_\delta$	0.0628	-0.5753	-0.1569	-0.2406	0.7767	0.0023	-0.0214	-0.0058	-0.0089	0.0288
	-0.5753	5.2738	1.4383	2.2054	-7.1196	-0.0214	0.1959	0.0534	0.0819	-0.2644
	-0.1569	1.4383	0.3923	0.6015	-1.9417	-0.0058	0.0534	0.0146	0.0223	-0.0721
	-0.2406	2.2054	0.6015	0.9223	-2.9773	-0.0089	0.0819	0.0223	0.0343	-0.1106
	0.7767	-7.1196	-1.9417	-2.9773	9.6115	0.0288	-0.2644	-0.0721	-0.1106	0.3570

Table VII

$$\left(\epsilon_i \sim t_{(n-1)},\ \phi_{ij} \sim t_{(n-1)},\ \delta_{ij} \sim t_{(n-1)}\right)$$

$\sigma_\epsilon^2 = 1.4076,\ \sigma_\phi^2 = 1.4076,\ \sigma_\delta^2 = 1.4076$

	$n = 25$					$n = 45$				
$b_\delta^{(2)}$	0.0094	-0.0862	-0.0235	-0.0361	0.1164	0.0017	-0.0153	-0.0042	-0.0064	0.0206
	-0.0862	0.7905	0.2156	0.3306	-1.0672	-0.0153	0.1402	0.0382	0.0586	-0.1893
	-0.0235	0.2156	0.0588	0.0902	-0.2910	-0.0042	0.0382	0.0104	0.0160	-0.0516
	-0.0361	0.3306	0.0902	0.1382	-0.4463	-0.0064	0.0586	0.0160	0.0245	-0.0791
	0.1164	-1.0672	-0.2910	-0.4463	1.4407	0.0206	-0.1893	-0.0516	-0.0791	0.2555
$b_\delta^{(3)}$	0.0034	-0.0315	-0.0086	-0.0132	0.0426	0.0016	-0.0143	-0.0039	-0.0060	0.0194
	-0.0315	0.2891	0.0788	0.1209	-0.3903	-0.0143	0.1315	0.0359	0.0550	-0.1776
	-0.0086	0.0788	0.0215	0.0330	-0.1064	-0.0039	0.0359	0.0098	0.0150	-0.0484
	-0.0132	0.1209	0.0330	0.0506	-0.1632	-0.0060	0.0550	0.0150	0.0230	-0.0743
	0.0426	-0.3903	-0.1064	-0.1632	0.5269	0.0194	-0.1776	-0.0484	-0.0743	0.2397
$b_\delta^{(4)}$	0.0527	-0.4832	-0.1318	-0.2021	0.6524	0.0023	-0.0206	-0.0056	-0.0086	0.0279
	-0.4832	4.4298	1.2081	1.8524	-5.9802	-0.0206	0.1893	0.0516	0.0791	-0.2555
	-0.1318	1.2081	0.3295	0.5052	-1.6310	-0.0056	0.0516	0.0141	0.0216	-0.0697
	-0.2021	1.8524	0.5052	0.7747	-2.5008	-0.0086	0.0791	0.0216	0.0331	-0.1068
	0.6524	-5.9802	-1.6310	-2.5008	8.0732	0.0279	-0.2555	-0.0697	-0.1068	0.3449

Table VIII

$\left(\epsilon_i \sim \text{Gamma}(5,\ \sqrt{\sigma_\epsilon^2/5}),\ \phi_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\phi^2/5}) \text{ and } \delta_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\delta^2/5})\right)$

$\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$										
	$n = 25$					$n = 45$				
$b_\delta^{(2)}$	0.0012	-0.0112	-0.0031	-0.0047	0.0151	0.0006	-0.0059	-0.0016	-0.0025	0.0079
	-0.0112	0.1028	0.0280	0.0430	-0.1388	-0.0059	0.0539	0.0147	0.0225	-0.0727
	-0.0031	0.0280	0.0076	0.0117	-0.0379	-0.0016	0.0147	0.0040	0.0061	-0.0198
	-0.0047	0.0430	0.0117	0.0180	-0.0580	-0.0025	0.0225	0.0061	0.0094	-0.0304
	0.0151	-0.1388	-0.0379	-0.0580	0.1874	0.0079	-0.0727	-0.0198	-0.0304	0.0982
$b_\delta^{(3)}$	0.0012	-0.0111	-0.0030	-0.0046	0.0149	0.0006	-0.0058	-0.0016	-0.0024	0.0079
	-0.0111	0.1013	0.0276	0.0424	-0.1368	-0.0058	0.0534	0.0146	0.0223	-0.0721
	-0.0030	0.0276	0.0075	0.0116	-0.0373	-0.0016	0.0146	0.0040	0.0061	-0.0197
	-0.0046	0.0424	0.0116	0.0177	-0.0572	-0.0024	0.0223	0.0061	0.0093	-0.0301
	0.0149	-0.1368	-0.0373	-0.0572	0.1847	0.0079	-0.0721	-0.0197	-0.0301	0.0973
$b_\delta^{(4)}$	0.0015	-0.0142	-0.0039	-0.0059	0.0191	0.0008	-0.0070	-0.0019	-0.0029	0.0094
	-0.0142	0.1299	0.0354	0.0543	-0.1754	-0.0070	0.0638	0.0174	0.0267	-0.0861
	-0.0039	0.0354	0.0097	0.0148	-0.0478	-0.0019	0.0174	0.0047	0.0073	-0.0235
	-0.0059	0.0543	0.0148	0.0227	-0.0733	-0.0029	0.0267	0.0073	0.0112	-0.0360
	0.0191	-0.1754	-0.0478	-0.0733	0.2367	0.0094	-0.0861	-0.0235	-0.0360	0.1162

Table IX

$$\left(\epsilon_i \sim \text{Gamma}(5,\ \sqrt{\sigma_\epsilon^2/5}),\ \phi_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\phi^2/5}) \text{ and } \delta_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\delta^2/5})\right)$$

$\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 1.0$										
	$n = 25$					$n = 45$				
$b_\delta^{(2)}$	0.0113	-0.1032	-0.0282	-0.0432	0.1394	0.0022	-0.0198	-0.0054	-0.0083	0.0268
	-0.1032	0.9464	0.2581	0.3958	-1.2776	-0.0198	0.1817	0.0496	0.0760	-0.2453
	-0.0282	0.2581	0.0704	0.1079	-0.3484	-0.0054	0.0496	0.0135	0.0207	-0.0669
	-0.0432	0.3958	0.1079	0.1655	-0.5343	-0.0083	0.0760	0.0207	0.0318	-0.1026
	0.1394	-1.2776	-0.3484	-0.5343	1.7247	0.0268	-0.2453	-0.0669	-0.1026	0.3311
$b_\delta^{(3)}$	0.0040	-0.0366	-0.0100	-0.0153	0.0494	0.0020	-0.0183	-0.0050	-0.0077	0.0248
	-0.0366	0.3356	0.0915	0.1403	-0.4530	-0.0183	0.1682	0.0459	0.0703	-0.2271
	-0.0100	0.0915	0.0250	0.0383	-0.1235	-0.0050	0.0459	0.0125	0.0192	-0.0619
	-0.0153	0.1403	0.0383	0.0587	-0.1894	-0.0077	0.0703	0.0192	0.0294	-0.0949
	0.0494	-0.4530	-0.1235	-0.1894	0.6116	0.0248	-0.2271	-0.0619	-0.0949	0.3065
$b_\delta^{(4)}$	0.0477	-0.4374	-0.1193	-0.1829	0.5905	0.0030	-0.0274	-0.0075	-0.0115	0.0370
	-0.4374	4.0099	1.0936	1.6769	-5.4134	-0.0274	0.2510	0.0685	0.1050	-0.3389
	-0.1193	1.0936	0.2983	0.4573	-1.4764	-0.0075	0.0685	0.0187	0.0286	-0.0924
	-0.1829	1.6769	0.4573	0.7012	-2.2638	-0.0115	0.1050	0.0286	0.0439	-0.1417
	0.5905	-5.4134	-1.4764	-2.2638	7.3081	0.0370	-0.3389	-0.0924	-0.1417	0.4575

Table X

$$\left(\epsilon_i \sim \text{Gamma}(5,\ \sqrt{\sigma_\epsilon^2/5}),\ \phi_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\phi^2/5})\text{ and }\delta_{ij} \sim \text{Gamma}(5,\ \sqrt{\sigma_\delta^2/5})\right)$$

$\sigma_\epsilon^2 = 1.0476,\ \sigma_\phi^2 = 1.0476,\ \sigma_\delta^2 = 1.0476$

	$n = 25$					$n = 45$				
$b_\delta^{(2)}$	0.0076	-0.0699	-0.0191	-0.0292	0.0944	0.0017	-0.0160	-0.0044	-0.0067	0.0216
	-0.0699	0.6410	0.1748	0.2681	-0.8654	-0.0160	0.1466	0.0400	0.0613	-0.1979
	-0.0191	0.1748	0.0477	0.0731	-0.2360	-0.0044	0.0400	0.0109	0.0167	-0.0540
	-0.0292	0.2681	0.0731	0.1121	-0.3619	-0.0067	0.0613	0.0167	0.0256	-0.0828
	0.0944	-0.8654	-0.2360	-0.3619	1.1683	0.0216	-0.1979	-0.0540	-0.0828	0.2672
$b_\delta^{(3)}$	0.0034	-0.0311	-0.0085	-0.0130	0.0419	0.0016	-0.0151	-0.0041	-0.0063	0.0204
	-0.0311	0.2848	0.0777	0.1191	-0.3844	-0.0151	0.1386	0.0378	0.0580	-0.1871
	-0.0085	0.0777	0.0212	0.0325	-0.1048	-0.0041	0.0378	0.0103	0.0158	-0.0510
	-0.0130	0.1191	0.0325	0.0498	-0.1608	-0.0063	0.0580	0.0158	0.0242	-0.0782
	0.0419	-0.3844	-0.1048	-0.1608	0.5190	0.0204	-0.1871	-0.0510	-0.0782	0.2526
$b_\delta^{(4)}$	0.0338	-0.3095	-0.0844	-0.1294	0.4179	0.0024	-0.0216	-0.0059	-0.0090	0.0291
	-0.3095	2.8373	0.7738	1.1865	-3.8304	-0.0216	0.1979	0.0540	0.0828	-0.2672
	-0.0844	0.7738	0.2110	0.3236	-1.0447	-0.0059	0.0540	0.0147	0.0226	-0.0729
	-0.1294	1.1865	0.3236	0.4962	-1.6018	-0.0090	0.0828	0.0226	0.0346	-0.1117
	0.4179	-3.8304	-1.0447	-1.6018	5.1710	0.0291	-0.2672	-0.0729	-0.1117	0.3607

It is observed from Table IV-X, that

- the variability of all the estimators $b_\delta^{(2)}$, $b_\delta^{(3)}$ and $b_\delta^{(4)}$ decrease as sample size increases.
- $b_\delta^{(3)}$ has the smallest variability.
- $b_\delta^{(4)}$ has the largest variability.
- $b_\delta^{(3)}$ is better than $b_\delta^{(2)}$ and $b_\delta^{(4)}$ both under the criterion of Loëwner ordering of empirical mean squared error matrices.
- $b_\delta^{(2)}$ is better than $b_\delta^{(4)}$ under the same criterion.
- variability increases as σ_δ^2 increases and vice versa.
- variability is not noticeably affected by change in the values of σ_ϵ^2 and σ_ϕ^2.

- the fluctuation in variability of $b_\delta^{(3)}$ due to change in sample size and values of various variances is minimum.
- the fluctuation in variability of $b_\delta^{(4)}$ is maximum and much higher.
- For Normal and t distributions, the variability of all three estimators are almost same, while they are quite less in case of Gamma distribution.
- the skewness of the distribution of the measurement errors has more impact on the variability of $b_\delta^{(2)}$ and $b_\delta^{(4)}$ in small samples. Such an effect decreases as sample size increases.
- the difference between theoretical and empirical values decreases drastically as sample size increases.

$b_\delta^{(3)}$ is "best" among all three estimators

variability is mainly affected by σ_δ^2 and $\gamma_{1\delta}$

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