A Structured Total Least Squares Algorithm for Approximate Greatest Common Divisors of Multivariate Polynomials

Lihong Zhi

Chinese Academy of Sciences

http://mmrc.iss.ac.cn/~lzhi

Joint work with Erich Kaltofen and Zhengfeng Yang

TLS-EIVM, Leuven, Belgium, August 21-23, 2006.

Approximate GCD Problem

Given polynomials $f_1, \ldots, f_s \in F[y_1, y_2, \ldots, y_r] \setminus \{0\}$, where F is $\mathbb{R}$ or $\mathbb{C}$; let $d_i = \text{tdeg}(f_i)$ and $k \leq d_i$ for all i with $1 \leq i \leq s$.
We wish to compute $\Delta f_1, \ldots, \Delta f_s \in F'[y_1, y_2, \ldots, y_r]$, where F' is $\mathbb{R}$ or $\mathbb{C}$, such that $\text{tdeg}(\Delta f_1) \leq d_1, \ldots, \text{tdeg}(\Delta f_s) \leq d_s$,

- $\text{tdeg}(\text{gcd}(f_1 + \Delta f_1, \ldots, f_s + \Delta f_s)) \geq k$,
- $\|\Delta f_1\|_2^2 + \cdots + \|\Delta f_s\|_2^2$ is minimized.

Approximate GCD Problem

Given polynomials $f_1, \ldots, f_s \in F[y_1, y_2, \ldots, y_r] \setminus \{0\}$, where F is $\mathbb{R}$ or $\mathbb{C}$; let $d_i = \operatorname{tdeg}(f_i)$ and $k \le d_i$ for all i with $1 \le i \le s$.

We wish to compute $\Delta f_1, \ldots, \Delta f_s \in F'[y_1, y_2, \ldots, y_r]$, where F' is $\mathbb{R}$ or $\mathbb{C}$, such that $\operatorname{tdeg}(\Delta f_1) \le d_1, \ldots, \operatorname{tdeg}(\Delta f_s) \le d_s$,

- $\operatorname{tdeg}(\gcd(f_1 + \Delta f_1, \ldots, f_s + \Delta f_s)) \ge k$,
- $\|\Delta f_1\|_2^2 + \cdots + \|\Delta f_s\|_2^2$ is minimized.

Problem depends on choice of norm $\|\cdot\|$, and notion of degree.
We use 2-norm on the coefficient vector, and total degree.

Previous Work on Approximate GCD

- Modified Euclidean algorithm for polynomials with floating point coefficients [Dunaway '74, Schönhage '85, Sasaki and Noda '89 & '91, Ochi et al. '91, Hribernig and Stetter '97, Beckermann and Labahn '98, Sasaki and Sasaki '01, Sanuki '05]
- Nearby roots matching, resultant-based algorithms, QR factorization, Hensel lifting strategy [Pan '01, Emiris et al. '97, Rupprecht '99, Zhi and Noda '01, Corless et al. '04]
- Least squares and SVD-based total least squares methods [Corless et al. '95, Chin et al. '98, Karmarkar and Lakshman '96 & '98, Zeng '03 & '04, Gao et al. '04, Diaz-Toca and Gonzalez-Vega '02 & '06]
- Structure preserving total least squares algorithms [Chu et al. '03, Li et al. '05, Kaltofen et al. '05 & '06, Botting et al. '05, Markovsky and Huffel '05, Winkler and Allan '06]

Globally Nearest Complex/Real Solution

Theorem 1. Let $f_1, \ldots, f_s \in F[y_1, \ldots, y_r] \setminus \{0\}$, where F is $\mathbb{C}$ or $\mathbb{R}$, $d_i = \mathrm{tdeg}(f_i)$ and $k \le d_i$ with $1 \le i \le s$. There exist $\widehat{f}_i \in F[y_1, y_2, \ldots, y_r], 1 \le i \le s$ with

$$\mathrm{tdeg}(\widehat{f}_i) \le d_i, \quad \text{and } \mathrm{tdeg}(\gcd(\widehat{f}_1, \ldots, \widehat{f}_s)) \ge k,$$

Globally Nearest Complex/Real Solution

Theorem 1. Let $f_1,\ldots,f_s \in F[y_1,\ldots,y_r]\setminus\{0\}$, where F is $\mathbb{C}$ or $\mathbb{R}$, $d_i = \mathrm{tdeg}(f_i)$ and $k \le d_i$ with $1 \le i \le s$. There exist $\widehat{f}_i \in F[y_1,y_2,\ldots,y_r], 1 \le i \le s$ with

$$\mathrm{tdeg}(\widehat{f}_i) \le d_i, \ \text{ and } \mathrm{tdeg}(\gcd(\widehat{f}_1,\ldots,\widehat{f}_s)) \ge k,$$

such that for all $\overline{f}_i \in F[y_1,y_2,\ldots,y_r], 1 \le i \le s$ with

$$\mathrm{tdeg}(\overline{f}_i) \le d_i, \text{ and } \mathrm{tdeg}(\gcd(\overline{f}_1,\ldots,\overline{f}_s)) \ge k,$$

Globally Nearest Complex/Real Solution

Theorem 1. Let $f_1,\ldots,f_s \in F[y_1,\ldots,y_r]\setminus\{0\}$, where F is $\mathbb{C}$ or $\mathbb{R}$, $d_i = \mathrm{tdeg}(f_i)$ and $k \le d_i$ with $1 \le i \le s$. There exist $\widehat{f}_i \in F[y_1,y_2,\ldots,y_r], 1 \le i \le s$ with

$$\mathrm{tdeg}(\widehat{f}_i) \le d_i, \text{ and } \mathrm{tdeg}(\gcd(\widehat{f}_1,\ldots,\widehat{f}_s)) \ge k,$$

such that for all $\overline{f}_i \in F[y_1,y_2,\ldots,y_r], 1 \le i \le s$ with

$$\mathrm{tdeg}(\overline{f}_i) \le d_i, \text{ and } \mathrm{tdeg}(\gcd(\overline{f}_1,\ldots,\overline{f}_s)) \ge k,$$

we have

$$\|\widehat{f}_1 - f_1\|_2^2 + \cdots + \|\widehat{f}_s - f_s\|_2^2 \le \|\overline{f} - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2.$$

Globally Nearest Complex/Real Solution

Theorem 1. Let $f_1, \ldots, f_s \in F[y_1, \ldots, y_r] \setminus \{0\}$, where F is $\mathbb{C}$ or $\mathbb{R}$, $d_i = \text{tdeg}(f_i)$ and $k \leq d_i$ with $1 \leq i \leq s$. There exist $\widehat{f}_i \in F[y_1, y_2, \ldots, y_r], 1 \leq i \leq s$ with

$$\text{tdeg}(\widehat{f}_i) \leq d_i, \ \text{ and } \text{tdeg}(\text{gcd}(\widehat{f}_1, \ldots, \widehat{f}_s)) \geq k,$$

such that for all $\overline{f}_i \in F[y_1, y_2, \ldots, y_r], 1 \leq i \leq s$ with

$$\text{tdeg}(\overline{f}_i) \leq d_i, \text{ and } \text{tdeg}(\text{gcd}(\overline{f}_1, \ldots, \overline{f}_s)) \geq k,$$

we have

$$\|\widehat{f}_1 - f_1\|_2^2 + \cdots + \|\widehat{f}_s - f_s\|_2^2 \leq \|\overline{f} - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2.$$

Remark: Theorem is false if $\text{tdeg}\,\text{gcd} = k$.

Generalized Sylvester Matrix

Lemma 1. Let $f_1, \ldots, f_s \in F[y_1, \ldots, y_r] \setminus \{0\}$, $d_i = \mathrm{tdeg}(f_i)$ and $k \leq d_i$ with $1 \leq i \leq s$. Then $\mathrm{tdeg}(\gcd(f_1, \ldots, f_s)) \geq k$ iff $\exists$ polynomials $u_1, \ldots, u_s \in F[y_1, \ldots, y_r]$ with

$$u_1 \neq 0, \quad \forall i, 2 \leq i \leq s\colon u_i f_1 + u_1 f_i = 0, \quad \mathrm{tdeg}(u_i) \leq d_i - k.$$

Equations give a linear system in the coefficients of $u_1, \ldots, u_s$.

Convolution Matrix

The convolution matrix $\mathrm{C}^{[l]}(f)$ produces the coefficient vector of $u \cdot f$ as $\mathrm{C}^{[l]}(f) \cdot \vec{u}$, where $l = \mathrm{tdeg}(u)$. For instance,

$$\mathrm{C}^{[2]}(a_2 y^2 + a_1 y + a_0) \cdot \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2 & 0 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{bmatrix} \cdot \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}.$$

In the univariate case, the matrix is of Toeplitz form. In the multivariate case, the dimensions of $\mathrm{C}^{[l]}(f)$ with $\mathrm{tdeg}(f) = m$ are $\binom{l+m+r}{r} \times \binom{l+r}{r}$.

Denote the coefficient matrix of the system $S_k(f_1, \ldots, f_s) =$

$$\begin{bmatrix} \mathrm{C}^{[d_2-k]}(f_1) & \mathbf{0} & \ldots & \mathbf{0} & \mathrm{C}^{[d_1-k]}(f_2) \\ \mathbf{0} & \mathrm{C}^{[d_3-k]}(f_1) & & \mathbf{0} & \mathrm{C}^{[d_1-k]}(f_3) \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathrm{C}^{[d_s-k]}(f_1) & \mathrm{C}^{[d_1-k]}(f_s) \end{bmatrix}$$

Denote the coefficient matrix of the system $S_k(f_1, \ldots, f_s) =$

$$\begin{bmatrix} \mathrm{C}^{[d_2-k]}(f_1) & \mathbf{0} & \cdots & \mathbf{0} & \mathrm{C}^{[d_1-k]}(f_2) \\ \mathbf{0} & \mathrm{C}^{[d_3-k]}(f_1) & & \mathbf{0} & \mathrm{C}^{[d_1-k]}(f_3) \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathrm{C}^{[d_s-k]}(f_1) & \mathrm{C}^{[d_1-k]}(f_s) \end{bmatrix}$$

Lemma 2. $\mathrm{tdeg}(\mathrm{gcd}(f_1, \ldots, f_s)) \geq k$ iff $S_k(f_1, \ldots, f_s)$ has rank deficiency at least one.

The Minimal Perturbation Problem

Our problem can be transformed into:

$$\min_{\text{tdeg}(\gcd(\overline{f}_1,\ldots,\overline{f}_s))\geq k} \|\overline{f}_1 - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2$$

$$\Longleftrightarrow \min_{\dim \text{Nullspace}(\overline{S}_k)\geq 1} \|\overline{f}_1 - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2$$

where $\overline{S}_k$ is the k-th Sylvester matrix generated by $\overline{f}_1,\ldots,\overline{f}_s$ with $\text{tdeg}\overline{f}_i \leq d_i, 1 \leq i \leq s$.

The Minimal Perturbation Problem

Our problem can be transformed into:

$$\min_{\text{tdeg}(\gcd(\overline{f}_1,\ldots,\overline{f}_s))\geq k} \|\overline{f}_1 - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2$$

$$\Longleftrightarrow \min_{\dim \text{Nullspace}(\overline{S}_k)\geq 1} \|\overline{f}_1 - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2$$

where $\overline{S}_k$ is the k-th Sylvester matrix generated by $\overline{f}_1, \ldots, \overline{f}_s$ with $\text{tdeg}\overline{f}_i \leq d_i, 1 \leq i \leq s$.

Nearest singular matrix + Sylvester structure

$\Longrightarrow$ Approximate GCD

Structure Preserving Low Rank Approximation

Let $S(\zeta) = [A_1(\zeta) \mid b(\zeta) \mid A_2(\zeta)]$ and let $A(\zeta) = [A_1(\zeta) \mid A_2(\zeta)]$, where ζ contains the coefficients of $f_1, \ldots, f_s$. We solve the structure-preserving total least norm problem

$$\min_{\mathbf{z} \in \mathbb{R}^\nu} \|\mathbf{z}\| \text{ or } \min_{\mathbf{z} \in \mathbb{C}^\nu} \|\mathbf{z}\| \quad \text{with} \quad A(\mathbf{c}+\mathbf{z})\mathbf{x} = b(\mathbf{c}+\mathbf{z})$$

for some vector $\mathbf{x}$. Here $\mathbf{c}$ is fixed to the initial coefficient vector.

Structure Preserving Low Rank Approximation

Let $S(\zeta) = [A_1(\zeta) \mid b(\zeta) \mid A_2(\zeta)]$ and let $A(\zeta) = [A_1(\zeta) \mid A_2(\zeta)]$, where ζ contains the coefficients of $f_1, \ldots, f_s$. We solve the structure-preserving total least norm problem

$$\min_{\mathbf{z}\in\mathbb{R}^\nu} \|\mathbf{z}\| \text{ or } \min_{\mathbf{z}\in\mathbb{C}^\nu} \|\mathbf{z}\| \quad \text{with} \quad A(\mathbf{c}+\mathbf{z})\mathbf{x} = b(\mathbf{c}+\mathbf{z})$$

for some vector $\mathbf{x}$. Here $\mathbf{c}$ is fixed to the initial coefficient vector.

Remark: We choose the column $b(\zeta)$ corresponding to the absolutely largest component in the first singular vector of $S(\zeta)$.

Example $r = 1, s = 2$.

$$S_k = [\mathbf{b}(\mathbf{c}), A(\mathbf{c})] = \begin{bmatrix} a_m & 0 & \cdots & 0 & 0 & b_n & 0 & \cdots & 0 & 0 \\ a_{m-1} & a_m & \cdots & 0 & 0 & b_{n-1} & b_n & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 & a_1 & 0 & 0 & \cdots & b_0 & b_1 \\ 0 & 0 & \cdots & 0 & a_0 & 0 & 0 & \cdots & 0 & b_0 \end{bmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{n-k+1} \quad \underbrace{\qquad\qquad\qquad}_{m-k+1}$$

Here $\mathbf{c}$ is fixed to the initial coefficient vector

$$\mathbf{c} = [a_m, \ldots, a_0, b_n, \ldots, b_0]^T.$$

Example (continued)

The structured real perturbations of S_k

$$\left[\begin{array}{cccccc} z_1 & & & z_{m+2} & & \\ z_2 & \ddots & & z_{m+3} & \ddots & \\ \vdots & \ddots & z_1 & \vdots & \ddots & z_{m+2} \\ z_{m+1} & & z_2 & z_{m+n+2} & & z_{m+3} \\ & \ddots & \vdots & & \ddots & \vdots \\ & & z_{m+1} & & & z_{m+n+2} \end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad}_{n-k+1} \quad \underbrace{\qquad\qquad\qquad}_{m-k+1}$$

can be represented by a vector $\mathbf{z} \in \mathbb{R}^{(m+n+2)\times 1}$:

$$\mathbf{z} = [z_1, z_2, \ldots, z_{m+n+1}, z_{m+n+2}]^T.$$

Initialization of $\mathbf{x}$ and $\Delta\mathbf{c}$

Suppose $H(\xi)\zeta = S(\zeta)\xi$ and $\mathbf{v}$ is the first singular vector of $S(\mathbf{c})$, then we compute $\Delta\mathbf{c}$ as:

$$\Delta\mathbf{c} = \mathbf{z} = -H(\mathbf{v})^T (H(\mathbf{v})H(\mathbf{v})^T)^{-1} S(\mathbf{c})\mathbf{v}.$$

We have $-S(\mathbf{z})\mathbf{v} = -H(\mathbf{v})\mathbf{z} = S(\mathbf{c})\mathbf{v}$, hence

$$S(\mathbf{c}+\mathbf{z})\mathbf{v} = 0.$$

Initialization of $\mathbf{x}$ and $\Delta\mathbf{c}$

Suppose $H(\xi)\zeta = S(\zeta)\xi$ and $\mathbf{v}$ is the first singular vector of $S(\mathbf{c})$, then we compute $\Delta\mathbf{c}$ as:

$$\Delta\mathbf{c} = \mathbf{z} = -H(\mathbf{v})^T(H(\mathbf{v})H(\mathbf{v})^T)^{-1}S(\mathbf{c})\mathbf{v}.$$

We have $-S(\mathbf{z})\mathbf{v} = -H(\mathbf{v})\mathbf{z} = S(\mathbf{c})\mathbf{v}$, hence

$$S(\mathbf{c}+\mathbf{z})\mathbf{v} = 0.$$

We initialize $\mathbf{x}$ as

$$\mathbf{x} = \left[-\frac{\mathbf{v}[1]}{\mathbf{v}[t]}, \ldots, -\frac{\mathbf{v}[t-1]}{\mathbf{v}[t]}, -\frac{\mathbf{v}[t+1]}{\mathbf{v}[t]}, \ldots\right]^T,$$

where $\mathbf{v}[t]$ is the absolutely largest component. We have

$$A(\mathbf{c}+\mathbf{z})\mathbf{x} = b(\mathbf{c}+\mathbf{z}).$$

Reference: Lemmerling'99

Lagrangian Multiplies [Lemmerling et al. '00]

We introduce the Lagrangian multiplies and define:

$$L(\mathbf{z},\mathbf{x},\lambda) = \frac{1}{2}\mathbf{z}^T\mathbf{z} - \lambda^T(b(\mathbf{c}+\mathbf{z}) - A(\mathbf{c}+\mathbf{z})\mathbf{x}).$$

Lagrangian Multiplies [Lemmerling et al. '00]

We introduce the Lagrangian multiplies and define:

$$L(\mathbf{z},\mathbf{x},\lambda) = \frac{1}{2}\mathbf{z}^T\mathbf{z} - \lambda^T(b(\mathbf{c}+\mathbf{z}) - A(\mathbf{c}+\mathbf{z})\mathbf{x}).$$

Apply the Newton method on the Lagrangian L yields:

$$\begin{bmatrix} W & J^T \\ J & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\mathbf{x} \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{g} + J^T\lambda \\ \mathbf{r}(\mathbf{z},\mathbf{x}) \end{bmatrix}$$

Lagrangian Multiplies [Lemmerling et al. '00]

We introduce the Lagrangian multiplies and define:

$$L(\mathbf{z}, \mathbf{x}, \lambda) = \frac{1}{2}\mathbf{z}^T\mathbf{z} - \lambda^T(b(\mathbf{c}+\mathbf{z}) - A(\mathbf{c}+\mathbf{z})\mathbf{x}).$$

Apply the Newton method on the Lagrangian L yields:

$$\begin{bmatrix} W & J^T \\ J & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\mathbf{x} \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{g} + J^T\lambda \\ \mathbf{r}(\mathbf{z}, \mathbf{x}) \end{bmatrix}$$

where $W = \begin{bmatrix} I_{t_1 \times t_1} & 0_{t_1 \times t_2} \\ 0_{t_2 \times t_1} & 0_{t_2 \times t_2} \end{bmatrix}, \quad t_1 = \sum_{i=1}^{s} \binom{d_i + r}{r},$

$t_2 = \sum_{i=1}^{s} \binom{d_i - k + r}{r} - 1, \quad \mathbf{g} = W \begin{bmatrix} \mathbf{z} \\ \mathbf{x} \end{bmatrix}, \quad J = [H, A(\mathbf{c}+\mathbf{z})]$. Here H is a Sylvester-like matrix: $H(\mathbf{x})\mathbf{z} = A(\mathbf{z})\mathbf{x} - b(\mathbf{z})$.

Complex Polynomials

Suppose $\mathbf{z} = \mathbf{z}_R + \boldsymbol{i}\mathbf{z}_I, \mathbf{x} = \mathbf{x}_R + \boldsymbol{i}\mathbf{x}_I$ and $\lambda = \lambda_R + \boldsymbol{i}\lambda_I$, we define

$$
\begin{aligned}
L(\mathbf{z},\mathbf{x},\lambda) &= \frac{1}{2}\mathbf{z}^H\mathbf{z} - \lambda_R^T(b(\mathbf{c}_R+\mathbf{z}_R) - A(\mathbf{c}_R+\mathbf{z}_R)\mathbf{x}_R + A(\mathbf{c}_I+\mathbf{z}_I)\mathbf{x}_I) \\
&\quad - \lambda_I^T(b(\mathbf{c}_I+\mathbf{z}_I) - A(\mathbf{c}_R+\mathbf{z}_R)\mathbf{x}_I - A(\mathbf{c}_I+\mathbf{z}_I)\mathbf{x}_R) \\
&= \frac{1}{2}\mathbf{z}^H\mathbf{z} + \lambda_R^T\mathbf{r}_R(\mathbf{z},\mathbf{x}) + \lambda_I^T\mathbf{r}_I(\mathbf{z},\mathbf{x}).
\end{aligned}
$$

First Order Iterative Update

We introduce a Sylvester-like matrix Y such that

$$Y(\mathbf{x})\mathbf{z} = A(\mathbf{z})\mathbf{x}.$$

Applying the Newton method on the Lagrangian L yields:

$$\begin{bmatrix} W & J^T \\ J & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z}_R \\ \Delta\mathbf{z}_I \\ \Delta\mathbf{x}_R \\ \Delta\mathbf{x}_I \\ \Delta\lambda_R \\ \Delta\lambda_I \end{bmatrix} = - \begin{bmatrix} \mathbf{g} + J^T \begin{bmatrix} \lambda_R \\ \lambda_I \end{bmatrix} \\ \mathbf{r}_R(\mathbf{z},\mathbf{x}) \\ \mathbf{r}_I(\mathbf{z},\mathbf{x}) \end{bmatrix},$$

Reference: H. Park et al. '99

Where

$$
\begin{aligned}
J &= \begin{bmatrix} H(\mathbf{x}_R) & -Y(\mathbf{x}_I) & A(\mathbf{c}_R+\mathbf{z}_R) & -A(\mathbf{c}_I+\mathbf{z}_I) \\ Y(\mathbf{x}_I) & H(\mathbf{x}_R) & A(\mathbf{c}_I+\mathbf{z}_I) & A(\mathbf{c}_R+\mathbf{z}_R) \end{bmatrix}, \\
W &= \begin{bmatrix} I_{t_1\times t_1} & 0_{t_1\times t_2} \\ 0_{t_2\times t_1} & 0_{t_2\times t_2} \end{bmatrix}, t_1 = 2\sum_{i=1}^{s}\binom{d_i+r}{r}, t_2 = -2+2\sum_{i=1}^{s}\binom{d_i-k+r}{r}, \\
\mathbf{g} &= W\begin{bmatrix} \mathbf{z}_R \\ \mathbf{z}_I \\ \mathbf{x}_R \\ \mathbf{x}_I \end{bmatrix}.
\end{aligned}
$$

Complex Optima for Real Inputs

Consider the polynomials $f = x^2 + 1$ and $g = x^2 + 2$.

Complex Optima for Real Inputs

Consider the polynomials $f = x^2 + 1$ and $g = x^2 + 2$.

The nearest pair of real polynomials with a common divisor

$$\overline{f} = 0.723598\,x^2 + 1.170810 \quad \text{and} \quad \overline{g} = 1.170822\,x^2 + 1.894436$$

The minimal perturbation $\|\overline{f} - f\|_2^2 + \|\overline{g} - g\|_2^2 = 0.145898.$

Complex Optima for Real Inputs

Consider the polynomials $f = x^2 + 1$ and $g = x^2 + 2$.

The nearest pair of real polynomials with a common divisor

$$\overline{f} = 0.723598\,x^2 + 1.170810 \quad \text{and} \quad \overline{g} = 1.170822\,x^2 + 1.894436$$

The minimal perturbation $\|\overline{f} - f\|_2^2 + \|\overline{g} - g\|_2^2 = 0.145898.$

The nearest pair of complex polynomials with a common root:

$$\begin{aligned} \overline{f} &= 0.81228x^2 - 0.14813\,\boldsymbol{i}\,x + 1.1169 \\ \overline{g} &= 1.1220x^2 + 0.096263\,\boldsymbol{i}\,x + 1.9240 \end{aligned}$$

The minimal perturbation $\|\overline{f} - f\|_2^2 + \|\overline{g} - g\|_2^2 = 0.1007615.$

Complex Optima for Real Inputs

Consider the polynomials $f = x^2 + 1$ and $g = x^2 + 2$.

The nearest pair of real polynomials with a common divisor

$\overline{f} = 0.723598x^2 + 1.170810 \quad \text{and} \quad \overline{g} = 1.170822x^2 + 1.894436$

The minimal perturbation $\|\overline{f} - f\|_2^2 + \|\overline{g} - g\|_2^2 = 0.145898.$

The nearest pair of complex polynomials with a common root:

$$\begin{aligned} \overline{f} &= 0.81228x^2 - 0.14813\boldsymbol{i}x + 1.1169 \\ \overline{g} &= 1.1220x^2 + 0.096263\boldsymbol{i}x + 1.9240 \end{aligned}$$

The minimal perturbation $\|\overline{f} - f\|_2^2 + \|\overline{g} - g\|_2^2 = 0.1007615.$

The complex conjugate is a second global optimum.

Random Complex Perturbation in the Initialization

We initialize $\mathbf{z}$ as:

$$\mathbf{z} = -H(\mathbf{v})^T (H(\mathbf{v}) H(\mathbf{v})^T)^{-1} S(\mathbf{c} + \boldsymbol{i} \Delta \mathbf{c}_{rand}) \mathbf{v},$$

where $\mathbf{v}$ is the first sing. vec. of $S(\mathbf{c} + \boldsymbol{i} \Delta \mathbf{c}_{rand})$, $\Delta \mathbf{c}_{rand}$ is a random real vector of small noise.

We initialize $\mathbf{x}$ by normalizing $\mathbf{v}$ as before, then

$$A(\mathbf{c} + \boldsymbol{i} \Delta \mathbf{c}_{rand} + \mathbf{z}) \mathbf{x} = b(\mathbf{c} + \boldsymbol{i} \Delta \mathbf{c}_{rand} + \mathbf{z}).$$

Multiple Local Minima

Consider the polynomials

$$f = 1000y^{10} + y^3 - 1 \quad \text{and} \quad g = y^2 - \frac{1}{100}.$$

We seek to compute the nearest pair of polynomials $\tilde{f}$ and $\tilde{g}$ that have a non-trivial GCD.

The algorithm converges after about ten iterations in average to the local minima:

$$0.0421579, 0.0463113, 0.0474087, 0.0493292, \ldots$$

for different initializations.

Among solutions, the polynomials

$$\begin{aligned} \tilde{f} &= 1000.0y^{10} + 0.0000147908y^9 + \cdots + 0.00415059y - 0.991601, \\ \tilde{g} &= 0.956139y^2 - 0.0887590y - 0.189618, \end{aligned}$$

have a common divisor

$$y - 0.4941547,$$

and the backward error is

$$\|f - \tilde{f}\|_2^2 + \|g - \tilde{g}\|_2^2 = 0.0421579.$$

It is the non-monic global minimum found by the global methods

Karmarkar and Lakshman Y.N. 1998; Hitz and Kaltofen 1998.

Structured Condition Number

The nearest unstructured singular matrix $\hat{A}$ has

$$\|S - \hat{A}\|_F^2 = 0.000000098975.$$

Structured Condition Number

The nearest unstructured singular matrix $\hat{A}$ has

$$\|S-\hat{A}\|_F^2 = 0.000000098975.$$

Let $\hat{S}$ be the nearest singular Sylvester matrix to S,

$$\begin{aligned}\|S-\hat{S}\|_F^2 &= 2\|f-\hat{f}\|_2^2 + 10\|g-\hat{g}\|_2^2 \\ &\geq 2\|f-\hat{f}\|_2^2 + 2\|g-\hat{g}\|_2^2 \\ &\geq 2\|f-\tilde{f}\|_2^2 + 2\|g-\tilde{g}\|_2^2 \geq 0.084315.\end{aligned}$$

Structured Condition Number

The nearest unstructured singular matrix $\hat{A}$ has

$$\|S-\hat{A}\|_F^2 = 0.000000098975.$$

Let $\hat{S}$ be the nearest singular Sylvester matrix to S,

$$\begin{aligned}\|S-\hat{S}\|_F^2 &= 2\|f-\hat{f}\|_2^2 + 10\|g-\hat{g}\|_2^2 \\ &\geq 2\|f-\hat{f}\|_2^2 + 2\|g-\hat{g}\|_2^2 \\ &\geq 2\|f-\tilde{f}\|_2^2 + 2\|g-\tilde{g}\|_2^2 \geq 0.084315.\end{aligned}$$

Sylvester $\neq$ Toeplitz à la Rump [2003]

Approximate GCD of Polynomials

Suppose $\overline{S}_k$ is the nearest singular generalized Sylvester matrix.

- Form the perturbed polynomials $\overline{f}_1, \ldots, \overline{f}_s$ from $\overline{S}_k$.

Approximate GCD of Polynomials

Suppose $\overline{S}_k$ is the nearest singular generalized Sylvester matrix.

- Form the perturbed polynomials $\overline{f}_1, \ldots, \overline{f}_s$ from $\overline{S}_k$.
- Suppose $\overline{k} = \mathrm{tdeg}(\gcd(\overline{f}_1, \ldots, \overline{f}_s))$.

Approximate GCD of Polynomials

Suppose $\overline{S}_k$ is the nearest singular generalized Sylvester matrix.

- Form the perturbed polynomials $\overline{f}_1, \ldots, \overline{f}_s$ from $\overline{S}_k$.
- Suppose $\overline{k} = \mathrm{tdeg}(\gcd(\overline{f}_1, \ldots, \overline{f}_s))$.
    - If $\dim \mathrm{Nullspace}(\overline{S}_k) = 1$, then set $\overline{k} = k$;

Approximate GCD of Polynomials

Suppose $\overline{S}_k$ is the nearest singular generalized Sylvester matrix.

- Form the perturbed polynomials $\overline{f}_1, \ldots, \overline{f}_s$ from $\overline{S}_k$.
- Suppose $\overline{k} = \text{tdeg}(\text{gcd}(\overline{f}_1, \ldots, \overline{f}_s))$.
  - If $\dim \text{Nullspace}(\overline{S}_k) = 1$, then set $\overline{k} = k$;
  - Otherwise, determine $\overline{k}$ from the rank deficiency of $\overline{S}$.

Approximate GCD of Polynomials

Suppose $\overline{S}_k$ is the nearest singular generalized Sylvester matrix.

- Form the perturbed polynomials $\overline{f}_1, \ldots, \overline{f}_s$ from $\overline{S}_k$.
- Suppose $\overline{k} = \text{tdeg}(\text{gcd}(\overline{f}_1, \ldots, \overline{f}_s))$.
  - If $\dim \text{Nullspace}(\overline{S}_k) = 1$, then set $\overline{k} = k$;
  - Otherwise, determine $\overline{k}$ from the rank deficiency of $\overline{S}$.
- Form polynomials $\overline{u}_1, \ldots, \overline{u}_s$ from the singular vector of $\overline{S}_{\overline{k}}$.

Approximate GCD of Polynomials

Suppose $\overline{S}_k$ is the nearest singular generalized Sylvester matrix.

- Form the perturbed polynomials $\overline{f}_1, \ldots, \overline{f}_s$ from $\overline{S}_k$.
- Suppose $\overline{k} = \text{tdeg}(\text{gcd}(\overline{f}_1, \ldots, \overline{f}_s))$.
  - If $\dim \text{Nullspace}(\overline{S}_k) = 1$, then set $\overline{k} = k$;
  - Otherwise, determine $\overline{k}$ from the rank deficiency of $\overline{S}$.
- Form polynomials $\overline{u}_1, \ldots, \overline{u}_s$ from the singular vector of $\overline{S}_{\overline{k}}$.
- Solve the linear system to obtain $d = \text{gcd}(\overline{f}_1, \ldots, \overline{f}_s)$

$$\begin{bmatrix} \mathrm{C}^{[\overline{k}]}(\overline{u}_1) \\ \mathrm{C}^{[\overline{k}]}(\overline{u}_2) \\ \vdots \\ \mathrm{C}^{[\overline{k}]}(\overline{u}_s) \end{bmatrix} \cdot \mathbf{d} = \begin{bmatrix} -\overline{\mathbf{f}}_1 \\ \overline{\mathbf{f}}_2 \\ \vdots \\ \overline{\mathbf{f}}_s \end{bmatrix}$$

Approx. GCD of Multivariate Polynomials

Ex.	d_i	k	e	*it.*	*error (Zeng)*	*error (GKMYZ)*	*error (STLS)*
1	7,7	4	3	4	2.44360e–4	2.59476e–4	6.50358e–5
2	7,7	4	5	1	2.44404e–8	2.59194e–8	6.50357e–9
3	7,7	4	7	1	2.44405e–12	2.59191e–12	6.50357e–13
4	7,7	4	9	1	2.44396e–16	2.59187e–16	6.50361e–17
5	6,6	3	2	5	2.26617	1.49524	4.80154e–1
6	10,10	5	4	4		2.74672e–3	1.84914e–3
7	8,8	4	5	2	7.09371e–5	2.38059e–5	2.01393e–5
8	40,40	30	5	2	1.39858e–3	4.83931e–4	4.39489e–4
9	10,9,8	5	3	4			6.21772e–2
10	8,7,8,6	4	5	2			4.04458e–6

Linearly Constrained Input Coefficients

- Monicity, sparsity:

$$\text{set} \quad z_j = \text{constant.}$$

Linearly Constrained Input Coefficients

- Monicity, sparsity:

$$\text{set} \quad z_j = \text{constant.}$$

- Nearest singular polynomials, approximate squarefree factorization, symmetry:

$$\text{set} \quad z_j = \lambda z_i$$

where λ is an integer.

Our Approach: Reduce Parameters

Let the linear constraints on the goal coefficients be

$$\Gamma\zeta = \gamma.$$

We construct for the linear system (**c** is the input coeff. vec.)

$$\Gamma\zeta = \gamma - \Gamma\mathbf{c}$$

a matrix C, a vector $\mathbf{d}$ and free parameters ζ^- such that

$$\zeta = C\zeta^- + \mathbf{d}, \quad \zeta^- = \begin{bmatrix} \zeta_{i_1} \\ \vdots \\ \zeta_{i_\mu} \end{bmatrix}; \quad \Gamma C = 0, \quad \Gamma\mathbf{d} = \gamma - \Gamma\mathbf{c}.$$

Our Approach: Reduce Parameters

Let the linear constraints on the goal coefficients be

$$\Gamma\zeta = \gamma.$$

We construct for the linear system (**c** is the input coeff. vec.)

$$\Gamma\zeta = \gamma - \Gamma\mathbf{c}$$

a matrix C, a vector **d** and free parameters ζ^- such that

$$\zeta = C\zeta^- + \mathbf{d}, \quad \zeta^- = \begin{bmatrix} \zeta_{i_1} \\ \vdots \\ \zeta_{i_\mu} \end{bmatrix}; \quad \Gamma C = 0, \quad \Gamma\mathbf{d} = \gamma - \Gamma\mathbf{c}.$$

Remark: We can add the constraints $\Gamma(\mathbf{c}+\mathbf{z}) = \gamma$ directly. However, the least squares problem has a larger dimension.

Initialization of $\mathbf{x}$ and $\mathbf{z}$

Let

$$\mathbf{z} = \mathbf{d} - C\,(H(\mathbf{v})C)^T (H(\mathbf{v})C\,(H(\mathbf{v})C)^T)^{-1} S(\mathbf{c}+\mathbf{d})\mathbf{v},$$

$\mathbf{v}$ is the first singular vector of the matrix $S(\mathbf{c}+\mathbf{d})$. We have

$$S(\mathbf{c}+\mathbf{z})\mathbf{v} = 0.$$

We initialize $\mathbf{x}$ as

$$\mathbf{x} = \left[-\frac{\mathbf{v}[1]}{\mathbf{v}[t]}, \ldots, -\frac{\mathbf{v}[t-1]}{\mathbf{v}[t]}, -\frac{\mathbf{v}[t+1]}{\mathbf{v}[t]}, \ldots \right]^T,$$

where $\mathbf{v}[t]$ is the absolutely largest component. We have

$$A(\mathbf{c}+\mathbf{z})\mathbf{x} = b(\mathbf{c}+\mathbf{z}).$$

First Order Iterative Update

Apply the Newton method on the Lagrangian L yields:

$$\begin{bmatrix} W_c & {J_c}^T \\ J_c & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z}^- \\ \Delta\mathbf{x} \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{g}_c + {J_c}^T \lambda \\ \mathbf{r}(\mathbf{z}, \mathbf{x}) \end{bmatrix},$$

First Order Iterative Update

Apply the Newton method on the Lagrangian L yields:

$$\begin{bmatrix} W_c & {J_c}^T \\ J_c & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{z}^- \\ \Delta \mathbf{x} \\ \Delta \lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{g}_c + {J_c}^T \lambda \\ \mathbf{r}(\mathbf{z}, \mathbf{x}) \end{bmatrix},$$

$$W_c = \begin{bmatrix} C^T C & 0 \\ 0 & 0 \end{bmatrix}, J_c = [HC, A(\mathbf{c} + \mathbf{z})], \mathbf{g}_c = \begin{bmatrix} C^T C \mathbf{z}^- + C^T \mathbf{d} \\ 0 \end{bmatrix}.$$

First Order Iterative Update

Apply the Newton method on the Lagrangian L yields:

$$\begin{bmatrix} W_c & {J_c}^T \\ J_c & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z}^- \\ \Delta\mathbf{x} \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{g}_c + {J_c}^T \lambda \\ \mathbf{r}(\mathbf{z},\mathbf{x}) \end{bmatrix},$$

$$W_c = \begin{bmatrix} C^T C & 0 \\ 0 & 0 \end{bmatrix}, J_c = [HC, A(\mathbf{c}+\mathbf{z})], \mathbf{g}_c = \begin{bmatrix} C^T C\mathbf{z}^- + C^T \mathbf{d} \\ 0 \end{bmatrix}.$$

The new coefficients are $\mathbf{c}+\mathbf{z}+\Delta\mathbf{z} = \mathbf{c}+\mathbf{z}+C\Delta\mathbf{z}^-$, satisfy

$$\Gamma(\mathbf{c}+\mathbf{z}+\Delta\mathbf{z}) = \Gamma(\mathbf{c}+\mathbf{z}) = \Gamma(\mathbf{c}+\mathbf{d}) = \gamma$$

throughout the iteration.

Nearest Singular Polynomials

Lemma 3. Let $f(y) \in F[y]$ with $\deg(f) = n$ over a field F of characteristic 0, and let k be a multiplicity with $2 \le k \le n$. Denote by $f^{[i]} = \mathrm{d}^i f/\mathrm{d}y^i$, then $\deg(\gcd(f^{[0]}, \ldots, f^{[k-1]})) \ge 1$ iff the matrix $S_k^{sing}(f) =$

$$\begin{bmatrix} \mathrm{C}^{[n-1]}(f^{[k-1]}) & \mathbf{0} & \cdots & \mathbf{0} & \mathrm{C}^{[n-k]}(f^{[0]}) \\ \mathbf{0} & \mathrm{C}^{[n-2]}(f^{[k-1]}) & & \mathbf{0} & \mathrm{C}^{[n-k]}(f^{[1]}) \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathrm{C}^{[n-k+1]}(f^{[k-1]}) & \mathrm{C}^{[n-k]}(f^{[k-2]}) \end{bmatrix}$$

has rank deficiency at least one.

Weighted Minimization Problem

For the nearest k-fold singular polynomial one optimizes

$$\|\Delta f\|,$$

while the GCD problem with the corresponding constraints on the coefficients optimizes

$$\sum \|\mathrm{d}^i \Delta f / \mathrm{d}y^i\|,$$

which has a different minimum.

We introduce a weight matrix D to the minimization problem,

$$\min_{\Delta\mathbf{c}\in\mathbb{R}^\nu} \|D\Delta\mathbf{c}\| \text{ or } \min_{\Delta\mathbf{c}\in\mathbb{C}^\nu} \|D\Delta\mathbf{c}\| \quad \text{with} \quad A(\mathbf{c}+\Delta\mathbf{c})\mathbf{x} = b(\mathbf{c}+\Delta\mathbf{c}).$$

Then the matrices become:

$$\begin{aligned} W_c &= \begin{bmatrix} C^T D^T D C & 0 \\ 0 & 0 \end{bmatrix}, \\ J_c &= [HC, A(\mathbf{c}+\mathbf{z})], \\ \mathbf{g}_c &= \begin{bmatrix} C^T D^T D C \mathbf{z}^- + C^T D^T \mathbf{d} \\ 0 \end{bmatrix}. \end{aligned}$$

Ex.	*d*	*k*	*it.*	error(ZNKW)	error(STLS)
1	4	2	11	.1763296120	.1763296120
		3	41	.6261127476	.6261127478
2	4	2	5	.1552760123e–12	.1552725104e–12
		3	4	.8834609009e–9	.9814622587e–9
		4	2	.2021848972e–4	.1958553174e–4
3	4	2	5	.1645037985e–10	.1645037985e–10
		3	5	.4144531274e–6	.4144531274e–6
		4	6	.1049993144	.1049993144
4	5	2	2	.2460987981e–8	.246098798e–8
	5	3	30	.3681785214	.3681785214
5	6	2	2	.3231668276e–5	.3231668277e–5
6	6	2	3	.3009788845e–11	.3009789179e–11
		3	7	.7453849284e–6	.7453849284e–6
		4	24	.4449023547	*failed*
7	5	2	8	.8565349347	.8565349347
8	21	2	3	.190477e–8	.18933576236e–8
		3	6	.963776e–4	*failed*

Univariate Singular Polynomials

Current Investigations

- Compute nearest factorization with a given degree pattern by applying STLS to Ruppert matrix.

Current Investigations

- Compute nearest factorization with a given degree pattern by applying STLS to Ruppert matrix.
- Compute nearest polynomial system with a given structure of zeros.

Current Investigations

- Compute nearest factorization with a given degree pattern by applying STLS to Ruppert matrix.

- Compute nearest polynomial system with a given structure of zeros.

- Nonlinear structure low rank approximation for Bezout matrix.

Current Investigations

- Compute nearest factorization with a given degree pattern by applying STLS to Ruppert matrix.
- Compute nearest polynomial system with a given structure of zeros.
- Nonlinear structure low rank approximation for Bezout matrix.
- Investigation of displacement operators for generalized Sylvester matrix or Bezout matrix or Ruppert matrix.

Current Investigations

- Compute nearest factorization with a given degree pattern by applying STLS to Ruppert matrix.
- Compute nearest polynomial system with a given structure of zeros.
- Nonlinear structure low rank approximation for Bezout matrix.
- Investigation of displacement operators for generalized Sylvester matrix or Bezout matrix or Ruppert matrix.
- How to certify a local minimum is a global minimum.

Code + Benchmarks at:

`http://mmrc.iss.ac.cn/~lzhi/Research/hybrid`

or

`google->kaltofen` (click on “Software”)

Thank you!