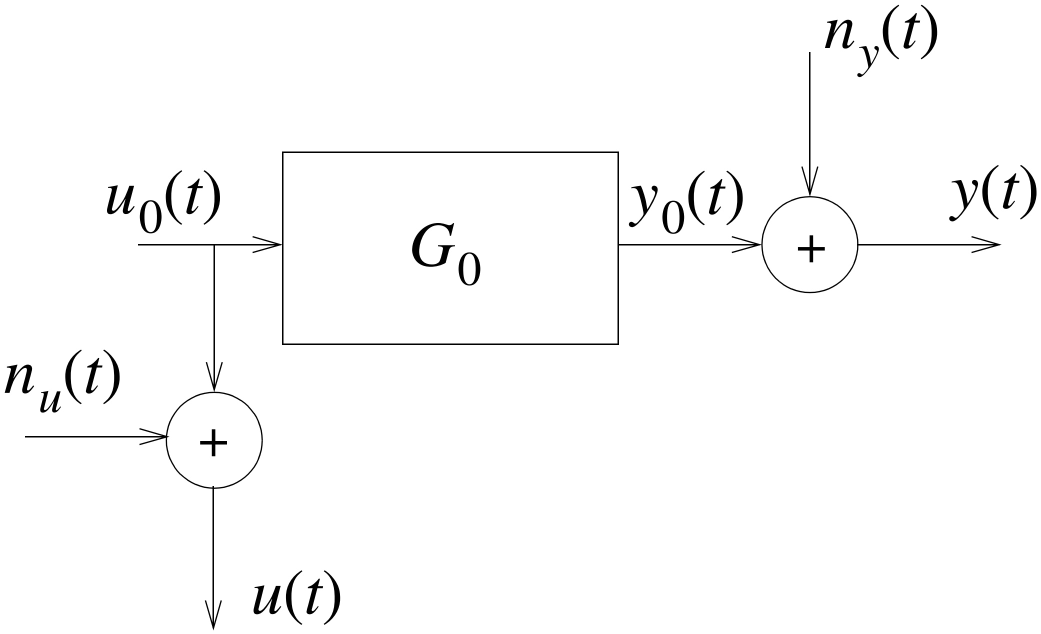
Identifiability analysis for errors-in-variables problems

J. Schoukens and R. Pintelon

Vrije universiteit Brussel

Problem statement



$$u(t) = u_0(t) + n_u(t)$$

$$y(t) = y_0(t) + n_y(t)$$

$$t = 1, \ldots, T$$

$u_0(t)$, $n_u(t)$, $n_y(t)$: filtered Gaussian noise

Frequency domain representation

Time to frequency domain transform

$$X(\Omega_k) = \frac{1}{\sqrt{N}}\sum_{t=0}^{T-1} x(t)e^{-2\pi kt/T}$$

$$U(\Omega_k) = U_0(\Omega_k) + N_u(\Omega_k)$$

$$Y(\Omega_k) = Y_0(\Omega_k) + N_y(\Omega_k)$$

$$k = 0, 1, \ldots, \frac{T}{2}$$

Problem statement (Cnt'd)

All information captured by the 2nd order moments

Asymptotically ($N \to \infty$), $\begin{bmatrix} U_0(\Omega_k) & N_u(\Omega_k) \\ Y_0(\Omega_k) & N_y(\Omega_k) \end{bmatrix}$ and $\begin{bmatrix} U_0(\Omega_l) & N_u(\Omega_l) \\ Y_0(\Omega_l) & N_y(\Omega_l) \end{bmatrix}$ independent for $k \neq l$

Asymptotically block diagonal matrix:

$$C(\Omega_k) = E\{ \begin{bmatrix} U(\Omega_k) & Y(\Omega_k) \end{bmatrix}^H \begin{bmatrix} U(\Omega_k) & Y(\Omega_k) \end{bmatrix} \}$$

$$= \begin{bmatrix} S_{u_0u_0}(\Omega_k) + S_{n_un_u}(\Omega_k) & G_0(\Omega_k) S_{u_0u_0}(\Omega_k) + S_{n_un_y}(\Omega_k) \\ \overline{G}_0(\Omega_k) S_{u_0u_0}(\Omega_k) + \overline{S_{n_un_y}}(\Omega_k) & |G_0(\Omega_k)|^2 S_{u_0u_0}(\Omega_k) + S_{n_yn_y}(\Omega_k) \end{bmatrix}$$

$$k = 0, 1, \ldots, \frac{T}{2}$$

Identifiability?

Can G_0 be identified from

$$C(\Omega_k) \qquad \{\Omega_k, k = 1, \ldots, N \leq T/2\}$$

Identifiability?

- parametric or nonparametric plant model

- parametric or nonparametric noise model

    + coloured, mutually correlated noise

    + coloured, mutually uncorrelated noise

    + white, mutually uncorrrelated noise

Basic idea

unknown parameters >< # constraints

$G_0(\Omega_k)$ a solution --> $\lambda_k G_0(\Omega_k)$ a solution?

Example 1

model: nonparametric plant $G(\Omega_k)$ and nonparametric noise $\begin{bmatrix} S_{n_u n_u}(\Omega_k) & 0 \\ 0 & S_{n_y n_y}(\Omega_k) \end{bmatrix}$

Available information

$$S_{u_0 u_0}(\Omega_k) + S_{n_u n_u}(\Omega_k) = \hat{S}_{u_0 u_0}(\Omega_k) + \hat{S}_{n_u n_u}(\Omega_k), \text{ (N real equations)}$$

$$G_0(\Omega_k) S_{u_0 u_0}(\Omega_k) = \left|\hat{G}(\Omega_k)\right| e^{j\angle\hat{G}(\Omega_k)} \hat{S}_{u_0 u_0}(\Omega_k), \text{ ($2N$ complex equations)}$$

$$\left|G_0(\Omega_k)\right|^2 S_{u_0 u_0}(\Omega_k) + S_{n_y n_y}(\Omega_k) = \left|\hat{G}(\Omega_k)\right|^2 \hat{S}_{u_0 u_0}(\Omega_k) + \hat{S}_{n_y n_y}(\Omega_k) \text{ (N real equations).}$$

Unknown parameters:

$$G(\Omega_k) : 2N \qquad S_{u_0 u_0}(\Omega_k) : N \qquad S_{n_u n_u}(\Omega_k) : N \qquad S_{n_y n_y}(\Omega_k) : N \;\rightarrow\; 5N$$

N degrees of freedom: check $G(\Omega_k) \rightarrow \lambda_k G(\Omega_k)$

Example 1 (cont'd)

$$\left|\hat{G}(\Omega_k)\right| = \lambda_k \left|G_0(\Omega_k)\right|$$

$$\hat{S}_{u_0u_0}(\Omega_k) = S_{u_0u_0}(\Omega_k)/\lambda_k$$

$$\hat{S}_{n_un_u}(\Omega_k) - S_{n_un_u}(\Omega_k) = S_{u_0u_0}(\Omega_k)(1 - 1/\lambda_k)$$

$$\hat{S}_{n_yn_y}(\Omega_k) - S_{n_yn_y}(\Omega_k) = \left|G_0(\Omega_k)\right|^2 S_{u_0u_0}(\Omega_k)(1 - \lambda_k)$$

Conclusion

$\hat{G}(\Omega_k) = \lambda_k G_0(\Omega_k)$, with $\lambda_{\min,\,k} \le \lambda_k \le \lambda_{\max,\,k}$

$$\lambda_{\min,\,k} = \frac{S_{u_0u_0}(\Omega_k)}{S_{n_un_u}(\Omega_k) + S_{u_0u_0}(\Omega_k)} \qquad \lambda_{\max,\,k} = \frac{\left|G_0(\Omega_k)\right|^2 S_{u_0u_0}(\Omega_k) + S_{n_yn_y}(\Omega_k)}{\left|G_0(\Omega_k)\right|^2 S_{u_0u_0}(\Omega_k)}$$

Example 2

model: parametric plant $G(\Omega_k, \theta_G)$ and nonparametric noise $\begin{bmatrix} S_{n_u n_u}(\Omega_k) & 0 \\ 0 & S_{n_y n_y}(\Omega_k) \end{bmatrix}$

Available information

$S_{u_0 u_0}(\Omega_k) + S_{n_u n_u}(\Omega_k) = \hat{S}_{u_0 u_0}(\Omega_k) + \hat{S}_{n_u n_u}(\Omega_k)$, ($N$ real equations)

$$G_0(\Omega_k) S_{u_0 u_0}(\Omega_k) = |\hat{G}(\Omega_k, \theta_G)| e^{j\angle \hat{G}(\Omega_k, \theta_G)} \hat{S}_{u_0 u_0}(\Omega_k),$$

$|G_0(\Omega_k)|^2 S_{u_0 u_0}(\Omega_k) + S_{n_y n_y}(\Omega_k) = |\hat{G}(\Omega_k, \theta_G)|^2 \hat{S}_{u_0 u_0}(\Omega_k) + \hat{S}_{n_y n_y}(\Omega_k)$ (N real equations).

Example 2 (Cnt'd)

model: parametric plant $G(\Omega_k, \theta_G)$ and nonparametric noise $\begin{bmatrix} S_{n_u n_u}(\Omega_k) & 0 \\ 0 & S_{n_y n_y}(\Omega_k) \end{bmatrix}$

$$G_0(\Omega_k)S_{u_0u_0}(\Omega_k) = |\hat{G}(\Omega_k, \theta_G)| e^{j\angle\hat{G}(\Omega_k, \theta_G)} \hat{S}_{u_0u_0}(\Omega_k),$$

$$|G(\Omega_k, \theta_G)S_{u_0u_0}(\Omega_k)| \to N$$

$$\text{phase } (G(\Omega_k, \theta_G)S_{u_0u_0}(\Omega_k)) = \text{phase } G(\Omega_k, \theta_G) \to n_{\theta_G} - 1$$

Assumption:

Plant model order: known
or
Plant has no quadrant symmetric poles or zeros

constraints:

$$3N + n_{\theta_G} - 1$$

Unknown parameters:

$$G(\Omega_k, \theta) : n_{\theta_G} \qquad S_{u_0 u_0}(\Omega_k) : N \qquad S_{n_u n_u}(\Omega_k) : N \qquad S_{n_y n_y}(\Omega_k) : N \;\rightarrow\; 3N + n_{\theta_G}$$

Conclusion: Parametric plant model $G(\Omega_k, \theta)$:

1 degree of freedom left:

$$\hat{G}(\Omega_k, \hat{\theta}_G) = \lambda G_0(\Omega_k),$$

with

$$\max \lambda_{\min, k} \le \lambda \le \min \lambda_{\max, k}$$

Example 3

model: parametric plant $G(\theta_G)$

parametric noise $\begin{bmatrix} S_{n_u n_u}(\theta_{n_u}) & 0 \\ 0 & S_{n_y n_y}(\theta_{n_y}) \end{bmatrix}$

non parametric or parametric signal model

Similar to previous situation: 1 degree of freedom left

$$\hat{S}_{u_0 u_0} = S_{u_0 u_0} / \lambda$$
$$\hat{S}_{n_u n_u} = S_{n_u n_u} + S_{u_0 u_0}(1 - 1/\lambda)$$
$$\hat{S}_{n_y n_y} = S_{n_y n_y} + |G|^2 S_{u_0 u_0}(1 - \lambda)$$

Possibility 1

Noise model order: known
Plant model order: known
Signal model: nonparametric

$$\hat{S}_{n_u n_u}(\theta_{n_u}) = S_{n_u n_u}(\theta_{n_u}) + S_{u_0 u_0}(\Omega_k)(1 - 1/\lambda)$$
$$\hat{S}_{n_y n_y}(\theta_{n_y}) = S_{n_y n_y}(\theta_{n_y}) + |G_0(\Omega_k)|^2 S_{u_0 u_0}(\Omega_k)(1 - \lambda) \tag{1}$$

Identifiable if for $\alpha \in \mathbb{R}_0$

i) the order of $S_{n_u n_u}(\theta_{n_u}) + \alpha S_{u_0 u_0}(\Omega_k)$ is larger than that of $S_{n_u n_u}(\theta_{n_u})$
or
ii) the order of $S_{n_y n_y}(\theta_{n_y}) + \alpha |G_0(\Omega_k)|^2 S_{u_0 u_0}(\Omega_k)$ is larger than that of $S_{n_y n_y}(\theta_{n_y})$

Possibility 2

Noise model order: known
Plant model order: not known
Signal model: non parametric

If the plant is known to have no quadrant symmetric poles or zeros --> orders can be retrieved

Previous situation is valid

Possibility 3

Noise model order: not known
Plant model order: not known
Signal model: parametric

1) If the plant is known to have no quadrant symmetric poles or zeros --> orders can be retrieved

2) Additional assumption

for $\alpha \in \mathbb{R}_0$

i) the order of $S_{n_u n_u}(\theta_{n_u}) + \alpha S_{u_0 u_0}(\Omega_k)$ is larger than that of $S_{n_u n_u}(\theta_{n_u})$

or

ii) the order of $S_{n_y n_y}(\theta_{n_y}) + \alpha |G_0(\Omega_k)|^2 S_{u_0 u_0}(\Omega_k)$ is larger than that of $S_{n_y n_y}(\theta_{n_y})$.

Illustration: Setup

plant model $G_0(z) = \dfrac{1 + 0.5z^{-1}}{0.8 - 12z^{-1} + 5.6z^{-2}}$

signal model $S_{u_0u_0} = |L(z)|^2, \quad L(z) = \dfrac{1}{1 - \dfrac{0.5}{\beta}z^{-1}}, \quad \beta = 1, 10, 100$

noise model

input: $S_{n_un_u} = 1$

output: $S_{n_yn_y} = |G_0(z)|^2$

model orders plant, signal, and noise are known

Illustration: results $\beta = 1$

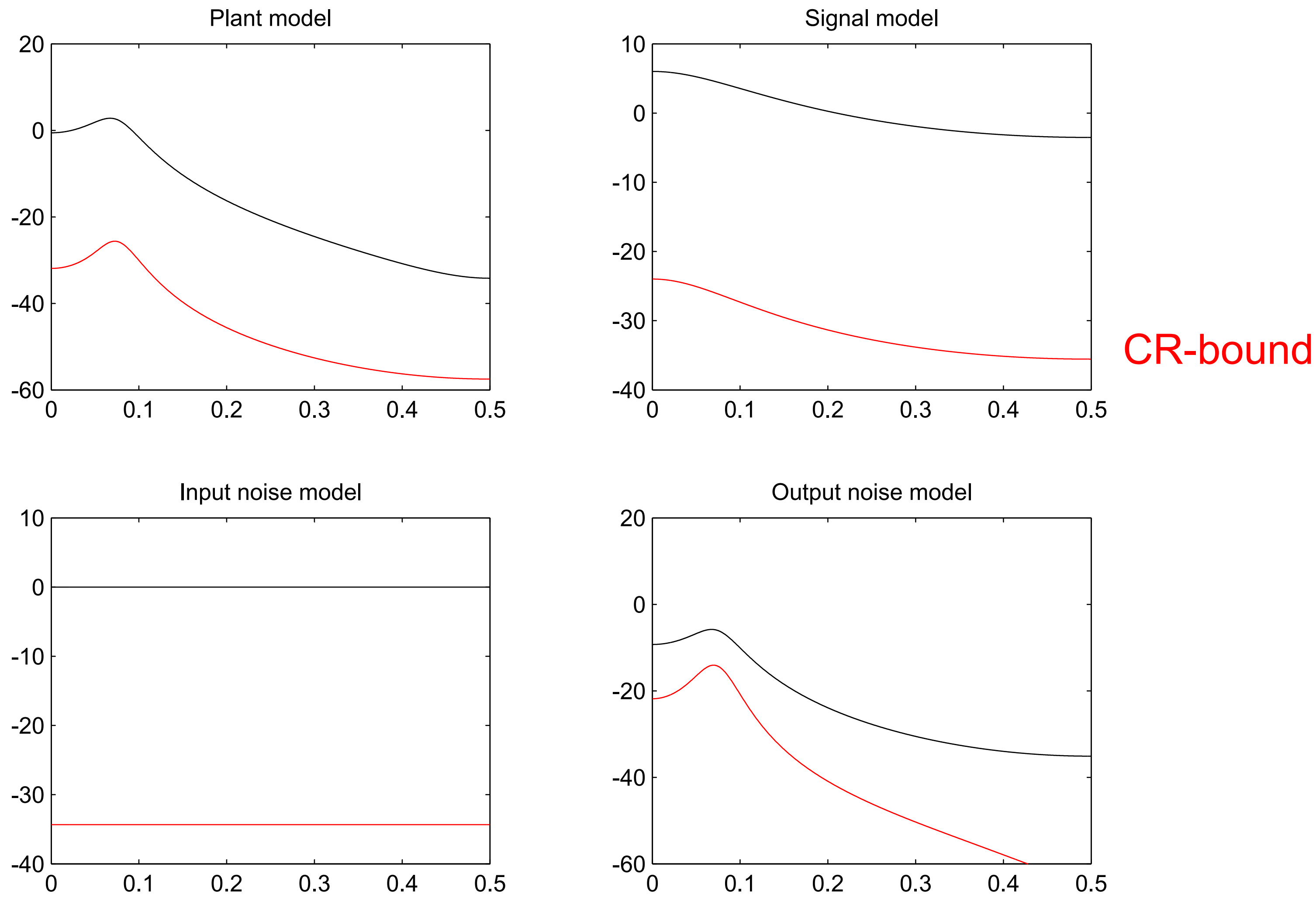


Illustration: results $\beta = 10$

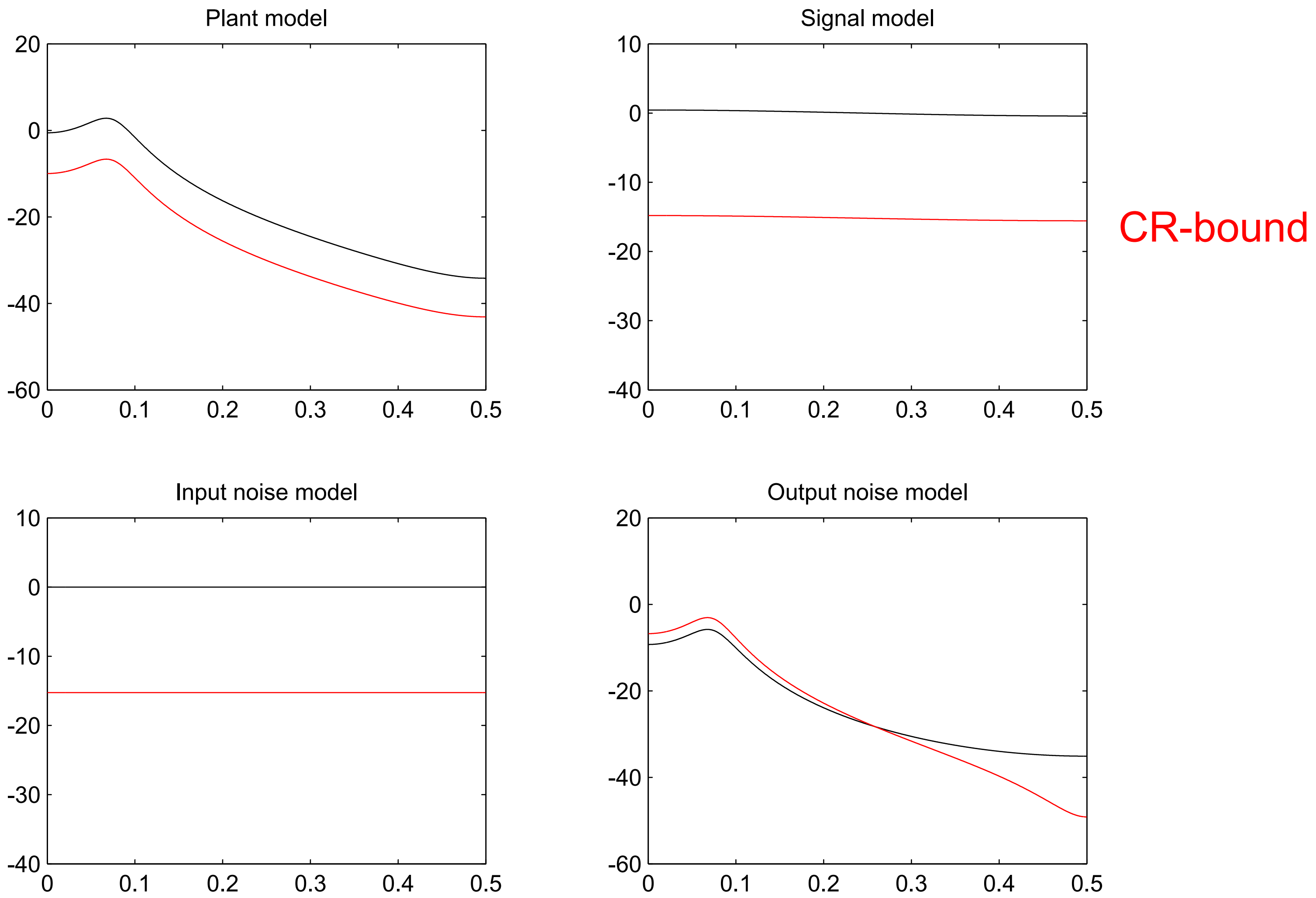


Illustration: results $\beta = 100$

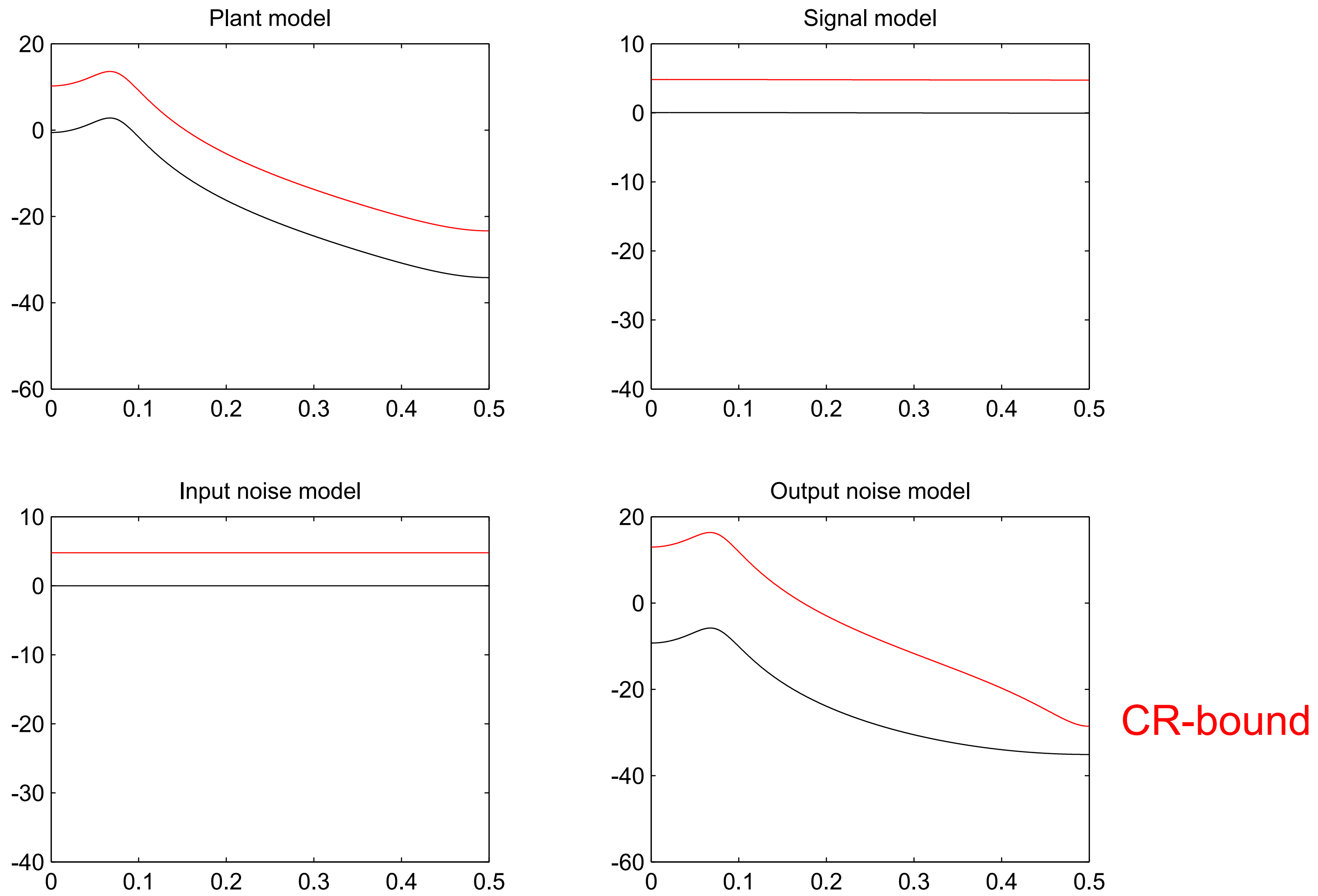
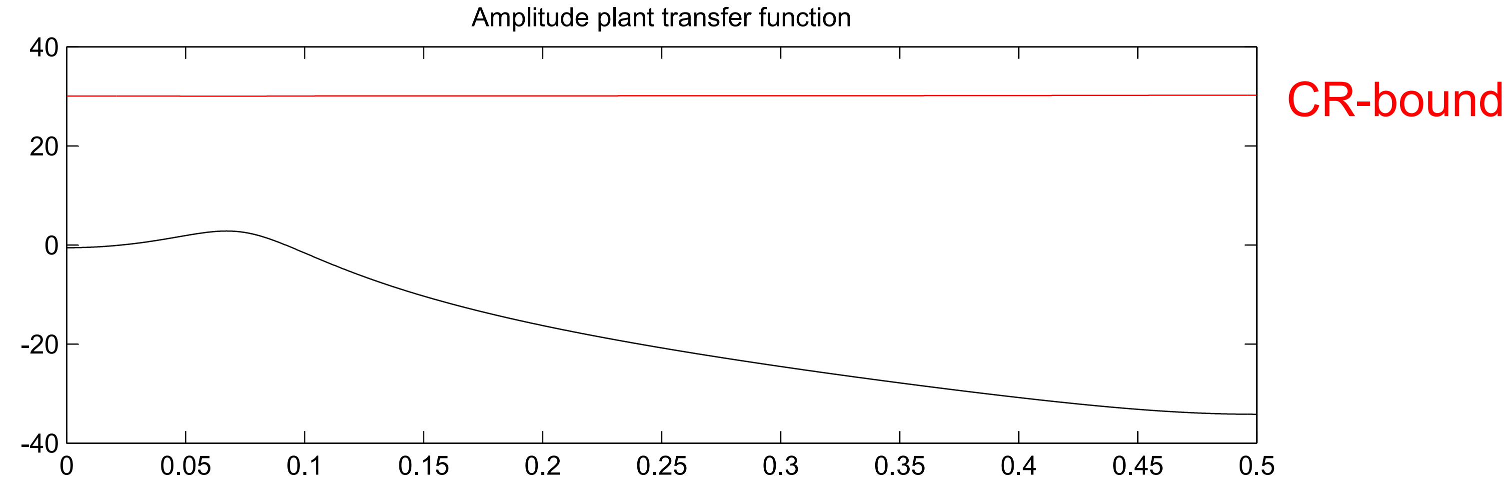
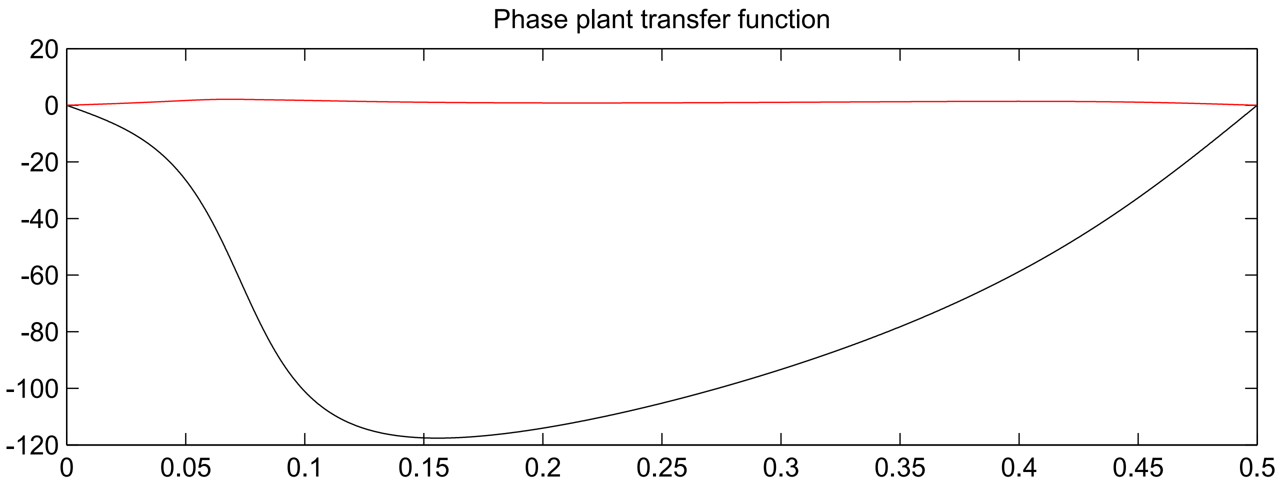
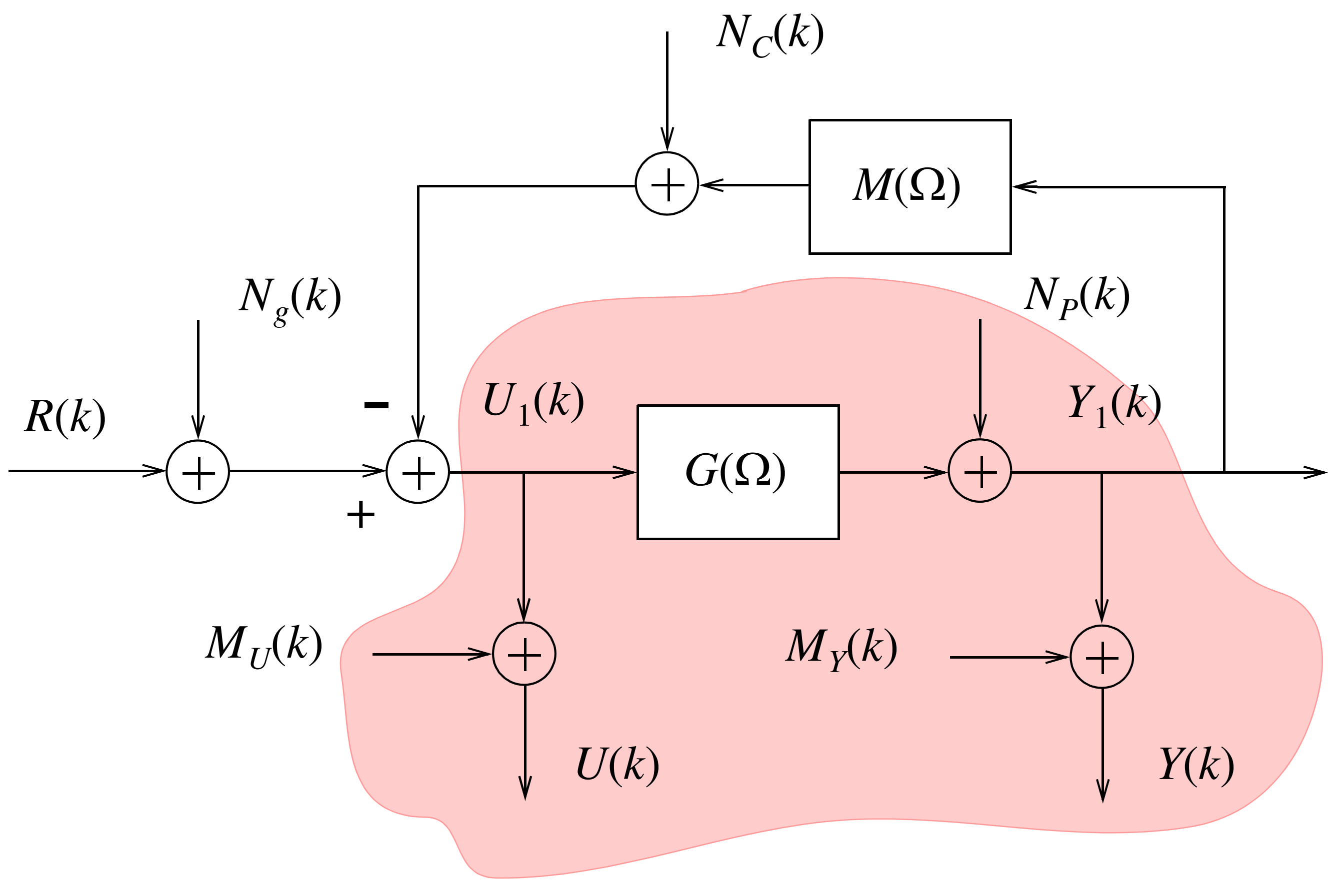


Illustration: results $\beta = 100$





Experimental constraint: periodic excitations



Nonparametric estimates for

$$S_{u_0u_0}(\Omega_k),\ S_{y_0u_0}(\Omega_k),\ S_{n_un_u}(\Omega_k),\ S_{n_yn_y}(\Omega_k),\ S_{n_un_y}(\Omega_k)$$

The problem is identifiable using nonparametric noise models.

Conclusions

- periodic signals: full blown feedback problem can be easily solved

- arbitrary signals:

    - simple tools for an identifiability study

    - a huge model selection problem is hidden